钱学森力学手稿

3

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著　　者 钱学森

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读者信箱:jdlgy@yahoo.cn

出 版 前 言

2011 年 12 月 11 日是西安交通大学杰出校友钱学森先生的百年诞辰。为缅怀钱学森学长，学习他的科学思想和卓越风范，展示其丰功伟绩和人格魅力，西安交通大学举办了“纪念钱学森诞辰 100 周年”系列活动：作为制片方之一，参与西部电影集团摄制传记故事片《钱学森》；与中央电视台合作，出品纪录片《实验班的故事——沿着钱学森走过的路》；扩建钱学森生平业绩展馆，向校内外开放；举办钱学森科学与教育思想研讨会；出版发行《钱学森力学手稿》、《钱学森年谱（初编）》、《钱学森第六次产业革命思想探微丛书》等。

钱学森先生在美国深造和工作期间留下大量珍贵手稿，这些手稿真实展示了钱学森先生博大精深的学识、开拓求实的精神和严谨奋进的作风，是钱老勇攀科学高峰和严谨治学的集中体现。这里，我们将部分原稿整理汇集成册，出版《钱学森力学手稿》，作为钱老百年诞辰的献礼。

《钱学森力学手稿》共 10 卷，包含两部分内容。第一部分是草稿，包括扁壳、球壳和圆柱壳屈曲分析的公式推导和数值演算。在研究圆柱壳轴压屈曲问题时，为了求得圆柱壳体的临界压力，在有关的五百多页草稿中，对多达二十多种可能的屈曲模

态逐一进行公式推演和数值计算，最终才找到满意的并在论文中采用的屈曲模态。仔细观察草稿中的数据列表，每个数字有效位数都长达八位，在手摇机械式计算机作为主要计算工具的年代，这串串数字凝聚着多少现今难以想象的艰辛劳动。

第二部分是手稿，以航空航天工程为核心，涵盖空气动力学、固体力学、火箭技术、工程控制论和物理力学等领域的部分学术论文手稿、打印稿和讲义。

《钱学森力学手稿》是在西安交通大学校领导的大力支持下，由西安交通大学航天航空学院沈亚鹏教授整理完成。图书出版过程中得到了西安交通大学党委宣传部、校友关系发展部、图书馆、航天航空学院等的积极协助，在此深表感谢。

Preliminary Calculation of Circular Cylinder (Ⅱ)

PART (III)

Effect of an Elliptic Hole

Ref Coker & Filon Photo-elasticity pp 540-544

If we write

$$\chi_1 = e^{2\xi} + \cos 2\eta$$

$$\chi_2 = e^{-2\xi} + \cos 2\eta$$

$$\chi_3 = e^{-2\xi} \cos 2\eta$$

$$\chi_4 = \xi$$

$$\chi_5 = e^{2\xi} \cos 2\eta$$

then it is found that the stress function can be written as

$$\chi = \frac{1}{16} T \left\{ \chi_1 + (2e^{2\alpha} - 1)\chi_2 - e^{4\alpha}\chi_3 + 4(1 - \cosh 2\alpha)\chi_4 - \chi_5 \right\}$$

Now the stresses given by the different stress functions are, if we write $(\cosh 2\xi - \cos 2\eta) = 2J^2$,

$$\begin{cases} 2J^4 \widehat{\xi\xi}_1 = \cos 4\eta - 4\cos 2\eta \cosh 2\xi + 2 + e^{4\xi} \\ 2J^4 \widehat{\eta\eta}_1 = \cos 4\eta - 4\cos 2\eta\, e^{2\xi} + 2 + e^{4\xi} \\ 2J^4 \widehat{\xi\eta}_1 = 2 \sin 2\eta \cosh 2\xi \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi}_2 = \cos 4\eta - 4\cos 2\eta \cosh 2\xi + 2 + e^{-4\xi} \\ 2J^4\,\widehat{\eta\eta}_2 = \cos 4\eta - 4\cos 2\eta\; e^{-2\xi} + 2 + e^{-4\xi} \\ 2J^4\,\widehat{\xi\eta}_2 = -2\sin 2\eta \cosh 2\xi \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi}_3 = \cos 4\eta \cdot e^{-2\xi} - \cos 2\eta\,(e^{-4\xi}+3) + 3e^{-2\xi} \\ 2J^4\,\widehat{\eta\eta}_3 = -\cos 4\eta \cdot e^{-2\xi} - 3e^{-2\xi} + \cos 2\eta\,(e^{-4\xi}+3) \\ 2J^4\,\widehat{\xi\eta}_3 = \sin 4\eta\, e^{-2\xi} - \sin 2\eta\,(e^{-4\xi}+3) \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi}_4 = \sinh 2\xi \\ 2J^4\,\widehat{\eta\eta}_4 = -\sinh 2\xi \\ 2J^4\,\widehat{\xi\eta}_4 = \sin 2\eta \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi}_5 = \cos 4\eta\, e^{2\xi} - \cos 2\eta\,(e^{4\xi}+3) + 3e^{2\xi} \\ 2J^4\,\widehat{\eta\eta}_5 = -\cos 4\eta\, e^{2\xi} + \cos 2\eta\,(e^{4\xi}+3) - 3e^{2\xi} \\ 2J^4\,\widehat{\eta\xi}_5 = -\sin 4\eta\, e^{2\xi} + \sin 2\eta\,(e^{4\xi}+3) \end{cases}$$

To find the strain energy increase in the specimen, it is best to find the increase in work done by the external forces, because the difficulty of carrying out the integration in elliptical coordinates.

We have

$$\left.\begin{cases} 2\mu J u_1 = (2-4\sigma)\,e^{2\xi} - (4-4\sigma)\cos 2\eta \\ 2\mu J v_1 = -(2-4\sigma)\sin 2\eta \end{cases}\right\}\ \text{due to } X_1$$

$$\left.\begin{cases} 2\mu J u_2 = (4-4\sigma)\cos 2\eta - (2-4\sigma)\,e^{-2\xi} \\ 2\mu J v_2 = -(2-4\sigma)\sin 2\eta \end{cases}\right\}\ \text{due to } X_2$$

$$\left.\begin{cases} 2\mu J u_3 = 2\,e^{-2\xi}\cos 2\eta \\ 2\mu J v_3 = 2\,e^{-2\xi}\sin 2\eta \end{cases}\right\}\ \text{due to } X_3$$

$$\{\ 2\mu J v_4 = -1 \qquad \text{due to } X_4$$

$$\left.\begin{cases} 2\mu J u_5 = -2\,e^{2\xi}\cos 2\eta \\ 2\mu J v_5 = 2\,e^{2\xi}\sin 2\eta \end{cases}\right\}\ \text{due to } X_5$$

Therefore the total displacements

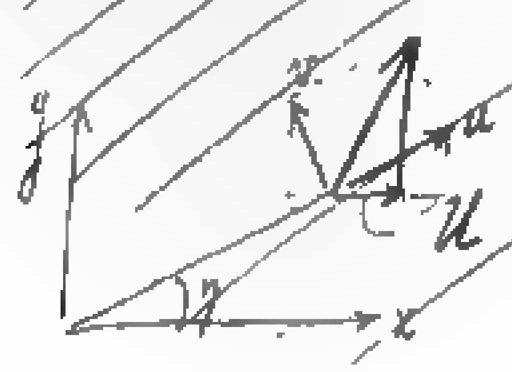
$$2\mu J u = \frac{1}{16} T \Big[(2-4\sigma) e^{2\xi} - (4-4\sigma) \cos 2\eta + (2e^{2\alpha}-1)\big\{ (4-4\sigma)\cos 2\eta - (2-4\sigma) e^{-2\xi} \big\}$$
$$- e^{4\alpha}\, 2 e^{-2\xi} \cos 2\eta + 4(1-\cosh 2\xi) + 2 e^{2\xi}\cos 2\eta \Big]$$

$$2\mu J v = \frac{1}{16} T \Big[-(2-4\sigma)\sin 2\eta - (2e^{2\alpha}-1)(2-4\sigma)\sin 2\eta\, 2e^{4\alpha} e^{-2\xi} \sin 2\eta$$
$$- 2 e^{2\xi}\sin 2\eta \Big]$$

or

$$\begin{cases} 2\mu J u = \frac{T}{8} \Big[(1-2\sigma) e^{2\xi} - (2-2\sigma)\cos 2\eta + (2e^{2\alpha}-1)\big\{ (2-2\sigma)\cos 2\eta - (1-2\sigma) e^{-2\xi} \big\} \\ \qquad - e^{4\alpha} e^{-2\xi}\cos 2\eta - 2(1-\cosh 2\xi) + e^{2\xi}\cos 2\eta \Big] \\ 2\mu J v = -\frac{T}{8} \sin 2\eta \Big[(1-2\sigma) + (2e^{2\alpha}-1)(1-2\sigma) + e^{4\alpha} e^{-2\xi} - e^{2\xi} \Big] \end{cases}$$

The component of displacement in the direction of tension



$$U = u\cos\eta - v\sin\eta$$

The uniform tension

$$\chi_0 = \frac{1}{2} T y^2 = \frac{T}{2} \sinh^2\xi \sin^2\eta$$

$$= \frac{T}{8} (\cosh 2\xi - 1)(1 - \cos 2\eta)$$

$$= \frac{T}{16} \left\{ (e^{2\xi} + \cos 2\eta) + (e^{-2\xi} + \cos 2\eta) - e^{2\xi}\cos 2\eta - e^{-2\xi}\cos 2\eta - 2 \right\}$$

$$= \frac{T}{16} \left\{ \chi_1 + \chi_2 - \chi_3 - \chi_5 \right\}$$

Thus the displacements

$$2\mu_0 J u_0 = \frac{T}{8} \left\{ 2(1-2\sigma)\sinh 2\xi + 2\sinh 2\xi \cos 2\eta \right\}$$

$$2\mu_0 J v_0 = -\frac{T}{8} \left\{ 2(1-2\sigma)\sin 2\eta + 2\cosh 2\xi \sin 2\eta \right\}$$

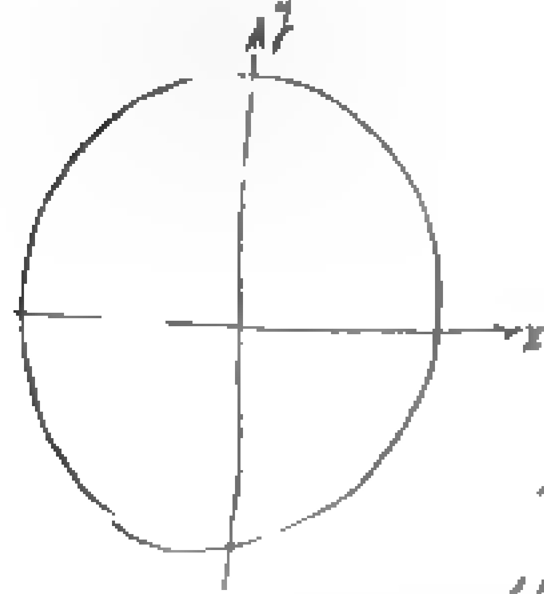
$$= -\frac{T}{8} \sin 2\eta \left\{ 2(1-2\sigma) + 2\cosh 2\xi \right\}$$

$$\cancel{T_\xi = T\cos\eta \qquad T_\eta = -T\sin\eta \quad \text{when } \xi = \alpha}$$

$$2\mu J(u - u_0) = \frac{T}{8} \left\{ (1-2\sigma)e^{-2\xi} - 2(1-\sigma)\cos 2\eta + (2e^{2\alpha} - 1)\left\{2(1-\sigma)\cos 2\eta - (1-2\sigma)e^{-2\xi}\right\} - e^{4\alpha}e^{-2\xi}\cos 2\eta - 2(1 - \cosh 2\alpha) + e^{-2\xi}\cos 2\eta \right\}$$

$$2\mu J(v - v_0) = -\frac{T}{8} \sin 2\eta \left\{ 2(e^{2\alpha} - 1) + e^{4\alpha}e^{-2\xi} - e^{-2\xi} \right\}$$

This encloses the circle at infinity



$$\sqrt{\frac{x^2+y^2}{c}} = \frac{c}{2} e^{\xi}$$

$$J^2 = \frac{1}{2}\frac{1}{2} e^{2\xi} c^2 \quad \therefore \quad v = \frac{c}{2} e^{\xi}$$

The [illegible] by [illegible] forces will be the [illegible] $\widehat{\xi\eta}$ & $\widehat{\xi\xi}$.

$$\widehat{\xi\eta} = -\frac{1}{2} T \sin 2\eta$$

$$\widehat{\xi\xi} = \frac{1}{2} T (1 + \cos 2\eta)$$

Increase in strain energy

$$= \frac{T}{32\mu} \frac{T}{2} \left[\int_0^{2\pi} (1+\cos 2\eta) \left\{ 4(1-\sigma)(e^{2\alpha}-1)\cos 2\eta - 2(1-\cosh 2\alpha) \right\} d\eta \right.$$

$$\left. + \int_0^{2\pi} \sin^2 2\eta \cdot 2(e^{2\alpha}-1)\, d\eta \right]$$

$$= \frac{T^2}{64\mu} \pi \left[4(1-\sigma)(e^{2\alpha}-1) + 4(\cosh 2\alpha - 1) + 2(e^{2\alpha}-1) \right]$$

$$= \frac{T^2}{32\mu} \pi \left[(3-2\sigma)(e^{2\alpha}-1) + 2(\cosh 2\alpha - 1) \right]$$

$$= \frac{T^2}{16\mu} \pi \left[(3-2\sigma) e^{\alpha} \sinh \alpha + (\cosh 2\alpha - 1) \right] c^2$$

Increase in strain energy $\mathcal{E}_{\theta}$

$$= \frac{(1+\sigma)T^2}{16E}\pi c^2 t\left[(3-2\sigma)(\sinh\alpha+\cosh\alpha)\sinh\alpha + (\cosh 2\alpha - 1)\right]$$

the axis of the ellipse,

$$a = c\cosh\alpha \qquad\qquad a^2-b^2=c^2$$
$$b = c\sinh\alpha$$

$$= t\,\frac{(1+\sigma)T^2}{16E}\pi\left[(3-2\sigma)(b+a)b + 2b^2\right]$$

$$\mathcal{E}_{\theta} = \frac{(1+\sigma)T^2\pi t}{16E}\left[(5-2\sigma)b^2 + (3-2\sigma)ab\right]$$

$$= \frac{(1+\sigma)T^2}{16E}(\pi ab)t\left[(3-2\sigma) + (5-2\sigma)\left(\frac{b}{a}\right)\right] \qquad \text{O.K.}$$

It is thus shown that the presence of a hole always increases the total strain energy, even compared with the whole plate plate. Therefore we have too much restraining a buckling stress can only be arrived by considering more accurately the interaction

Now for the sake of simplicity, go back to the case of a circular buckled region. Here, in order that the buckled circular plate be clamp supported, we choose the form of buckling to be

$$\left(\frac{w}{R}\right)_0 = \frac{\left(\frac{r}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2}$$

$$\left(\frac{w}{R}\right) = \frac{\left(\frac{r}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2} - f\left\{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2\right\}^2$$

Then

$$\begin{cases} \frac{1}{R}\frac{\partial w}{\partial \theta} = -\frac{1}{2}\left(\frac{r}{R}\right)^2 \sin 2\theta \\ \frac{1}{R}\frac{\partial w_0}{\partial \theta} = -\frac{1}{2}\left(\frac{r}{R}\right)^2 \sin 2\theta \end{cases}$$

$$\frac{1}{R}\frac{\partial^2 w}{\partial r \partial \theta} = \frac{1}{R}\frac{\partial^2 w_0}{\partial r \partial \theta} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin 2\theta$$

$$\frac{1}{R}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{R}\frac{\partial^2 w_0}{\partial \theta^2} = -\left(\frac{r}{R}\right)^2 \cos 2\theta$$

$$\begin{cases} \frac{1}{R}\frac{\partial w}{\partial r} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin^2\theta + 4f\left\{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2\right\}\left(\frac{r}{R}\right)^2 \frac{1}{r} \\ \frac{1}{R}\frac{\partial w_0}{\partial r} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin^2\theta \end{cases}$$

$$\begin{cases} \frac{1}{R}\frac{\partial^2 w}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta + 4f\left\{\left(\frac{a}{R}\right)^2 - 3\left(\frac{r}{R}\right)^2\right\}\frac{1}{R^2} \\ \frac{1}{R}\frac{\partial^2 w_0}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta \end{cases}$$

$$\frac{1}{r^4}\left(\frac{\partial \varphi}{\partial \theta}\right)^2 - \frac{1}{r^4}\left(\frac{\partial w}{\partial \theta}\right)^2 = 0$$

$$\frac{1}{r^2}\left(\frac{\partial^2 \varphi}{\partial r \partial \theta}\right)^2 - \frac{1}{r^2}\left(\frac{\partial^2 w}{\partial r \partial \theta}\right)^2 = 0$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\,\frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w}{\partial r}\,\frac{\partial^2 w}{\partial r^2}\right\} \text{[illegible]}$$

$$= \frac{1}{R^2}(\cos^2\theta)^2 - \frac{1}{R^2}\left[\cos^2\theta - 4f\left(\frac{a^2}{R^2} - \frac{a^2}{R^2}\right)\right]\left[\cos^2\theta - 4f\left(\frac{a^2}{R^2} - 3\frac{a^2}{R^2}\right)\right]$$

$$= \frac{1}{R^2}\left[8f\left(\frac{a^2}{R^2} - 2\frac{a^2}{R^2}\right)\cos^2\theta - 16f^2\left(\frac{a^2}{r^2} - \frac{a^2}{r^2}\right)\left(\frac{a^2}{r^2} - 3\frac{a^2}{R^2}\right)\right]$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\,\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\,\frac{\partial^2 w_0}{\partial \theta^2}\right\}$$

$$= \frac{1}{R^2}\cos 2\theta \cdot 4f\left\{\frac{a^2}{r^2} - 3\frac{a^2}{R^2}\right\}$$

$$\nabla^4 \varphi = \frac{4Ef}{R^2}\left[2\left(\frac{a^2}{R^2} - 2\frac{a^2}{r^2}\right)\sin^2\theta + \cos 2\theta\left(\frac{a^2}{R^2} - 3\frac{a^2}{R^2}\right) - 4f\left(\frac{a^2}{r^2} - \frac{a^2}{R^2}\right)\left(\frac{a^2}{R^2} - 3\frac{a^2}{R^2}\right)\right]$$

$$= \frac{4Ef}{R^2}\left[\left(\frac{a^2}{R^2} - 2\frac{a^2}{R^2}\right) - \left(\frac{a^2}{R^2}\right)\cos 2\theta - 4f\left(\frac{a^2}{r^2} - \frac{a^2}{r^2}\right)\left(\frac{a^2}{R^2} - 3\frac{a^2}{R^2}\right)\right]$$

If [illegible] $\left(\frac{a}{R}\right) = \frac{\text{[illegible]}}{r^2}$ [illegible]

010 [illegible]

$$\nabla^4\varphi = C\left[p^2-4\right]\left[(p-2)^2-4\right] r^{p-4}\cos 2\theta$$

$p=6,$ $$\nabla^4\varphi = C\cdot 32\cdot 12\, r^2\cos 2\theta$$

$$\therefore \qquad C = \frac{-K}{384}$$

$$\therefore \qquad \varphi_4 = -\frac{Ef}{384R^2}\frac{r^6}{R^2}\cos 2\theta$$

Therefore the particular integral is

$$\boxed{\varphi_0 = \frac{EfR^2}{64}\left[\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\frac{r^4}{R^4} + \frac{2}{9}\left(2f\frac{a^2}{R^2}-1\right)\frac{r^6}{R^6} - \frac{1}{6}\frac{r^6}{R^6}\cos 2\theta - \frac{1}{12}f\frac{r^8}{R^8}\right]}$$

Due to the symmetry of this problem, the solution of the homogeneous equation

$$\nabla^4\varphi = 0$$

can be written as

$$\frac{\varphi_1}{R^2} = \left(\frac{Ef}{64}\right)\Big\{ + Q_0\frac{r^2}{R^2} + S_0 + \cos 2\theta\left[P_2\left(\frac{r}{R}\right)^2 + R_2\left(\frac{r}{R}\right)^4 \dots\right]$$
$$+ \cos 4\theta\left[P_4\left(\frac{r}{R}\right)^4 + R_4\left(\frac{r}{R}\right)^6\right] + \cos 6\theta\left[P_6\left(\frac{r}{R}\right)^6 + R_6\left(\frac{r}{R}\right)^8\right]\Big\}$$

$$\frac{\varphi}{R^2} = \left(\frac{Ef}{64}\right)\Bigg[\left\{\Phi + \frac{1}{4}Q_0\left(\frac{r}{R}\right)^2 + \left(\frac{r}{R}\right)^2\left(1 - f\frac{r^2}{R^2}\right)\left(\frac{r}{R}\right)^4 + \frac{2}{9}\left(2f\frac{r^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^6 - \frac{1}{12}f\left(\frac{r}{R}\right)^8\right\}$$

$$+ \cos 2\theta\left\{P_2\left(\frac{r}{R}\right)^2 + R_2\left(\frac{r}{R}\right)^4 + \quad - \frac{1}{6}\left(\frac{r}{R}\right)^6\right\}$$

$$+ \cos 4\theta\left\{P_4\left(\frac{r}{R}\right)^4 + R_4\left(\frac{r}{R}\right)^6\right\}$$

$$+ \cos 6\theta\left\{P_6\left(\frac{r}{R}\right)^6 + R_6\left(\frac{r}{R}\right)^8\right\}$$

$$\frac{1}{r}\frac{\partial\varphi}{\partial r} = \left(\frac{Ef}{64}\right)\Bigg[\left\{\frac{1}{2}Q_0 + 4\left(\frac{r}{R}\right)^2\left(1 - f\frac{r^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{4}{3}\left(2f\frac{r^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{2}{3}f\left(\frac{r}{R}\right)^6\right\}$$

$$+ \cos 2\theta\left\{2P_2 + 4R_2\left(\frac{r}{R}\right)^2 - \left(\frac{r}{R}\right)^4\right\}$$

$$+ \cos 4\theta\left\{4P_4\left(\frac{r}{R}\right)^2 + 6R_4\left(\frac{r}{R}\right)^4\right\}$$

$$+ \cos 6\theta\left\{6P_6\left(\frac{r}{R}\right)^4 + 8R_6\left(\frac{r}{R}\right)^6\right\}$$

$$\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = \frac{Ef}{64}\Bigg[-4\cos 2\theta\left\{P_2 + R_2\left(\frac{r}{R}\right)^2 - \frac{1}{6}\left(\frac{r}{R}\right)^4\right\}$$

$$- 16\cos 4\theta\left\{P_4\left(\frac{r}{R}\right)^2 + R_4\left(\frac{r}{R}\right)^4\right\}$$

$$- 36\cos 6\theta\left\{P_6\left(\frac{r}{R}\right)^4 + R_6\left(\frac{r}{R}\right)^6\right\}\Bigg]$$

$$\widehat{rr} = \frac{Ef}{64}\Big[\Big\{\tfrac{1}{2}Q_0 + 4\tfrac{a^2}{R^2}\Big(1 - f\tfrac{a^2}{R^2}\Big)\Big(\tfrac{r}{R}\Big)^2 + \tfrac{4}{3}\Big(2f\tfrac{a^2}{R^2} - 1\Big)\Big(\tfrac{r}{R}\Big)^4 - \tfrac{2}{3}f\Big(\tfrac{r}{R}\Big)^6\Big\}$$

$$- \cos 2\theta\Big\{2P_2 + \tfrac{1}{3}\Big(\tfrac{r}{R}\Big)^4\Big\}$$

$$- \cos 4\theta\Big\{12 P_4\Big(\tfrac{r}{R}\Big)^2 + 10 P_4\Big(\tfrac{r}{R}\Big)^4\Big\}$$

$$- \cos 6\theta\Big\{30 P_6\Big(\tfrac{r}{R}\Big)^4 + 28 R_6\Big(\tfrac{r}{R}\Big)^6\Big\}\Big]$$

$$\widehat{\theta\theta} = \frac{Ef}{64}\Big[\Big\{\tfrac{1}{2}Q_0 + 12\tfrac{a^2}{R^2}\Big(1 - f\tfrac{a^2}{R^2}\Big)\Big(\tfrac{r}{R}\Big)^2 + \tfrac{20}{3}\Big(2f\tfrac{a^2}{R^2} - 1\Big)\Big(\tfrac{r}{R}\Big)^4 - \tfrac{14}{3}f\Big(\tfrac{r}{R}\Big)^6\Big\}$$

$$+ \cos 2\theta\Big\{P_2 + 12 P_2\Big(\tfrac{r}{R}\Big)^2 - 5\Big(\tfrac{r}{R}\Big)^4\Big\} \qquad !!!$$

$$+ \cos 4\theta\Big\{12 P_4\Big(\tfrac{r}{R}\Big)^2 + 30 P_4\Big(\tfrac{r}{R}\Big)^4\Big\}$$

$$+ \cos 6\theta\Big\{30 P_6\Big(\tfrac{r}{R}\Big)^4 + 56 R_6\Big(\tfrac{r}{R}\Big)^6\Big\}$$

$$\widehat{r\theta} = \Big(\frac{Ef}{64}\Big)\Big[2\sin 2\theta\Big\{P_2 + 3P_2\Big(\tfrac{r}{R}\Big)^2 - \tfrac{5}{6}\Big(\tfrac{r}{R}\Big)^4\Big\}$$

$$+ 4\sin 4\theta\Big\{3P_4\Big(\tfrac{r}{R}\Big)^2 + 5P_4\Big(\tfrac{r}{R}\Big)^4\Big\}$$

$$+ 6\sin 6\theta\Big\{5P_6\Big(\tfrac{r}{R}\Big)^4 + 7R_6\Big(\tfrac{r}{R}\Big)^6\Big\}\Big]$$

$$\frac{\partial u}{\partial r} = \frac{1}{E}\left(\widehat{rr} - \nu\,\widehat{\theta\theta}\right)$$

$$\frac{u}{R} = \frac{a}{6\mu}\Big[\Big\{(1-\nu)\tfrac{1}{2}Q_0\left(\tfrac{a}{r}\right) + \tfrac{4}{3}(1-3\nu)\tfrac{a^2}{r^2}\left(1-2\tfrac{a^2}{r^2}\right)\left(\tfrac{a}{r}\right)^3$$

$$+ \frac{4}{15}(1-5\nu)\left(3f\frac{a^2}{r^2}-1\right)\left(\frac{a}{r}\right)^5 - \frac{2}{21}(1-7\nu)f\left(\frac{a}{r}\right)^7\Big\}$$

$$- \cos 2\theta\Big\{(2+\nu)P_2\left(\tfrac{a}{r}\right) + 4\nu P_2\left(\tfrac{a}{r}\right)^3 + \tfrac{4}{15}(1-15\nu)\left(\tfrac{a}{r}\right)^5\Big\}$$

$$- \cos 4\theta\Big\{4(1+\nu)P_4\left(\tfrac{a}{r}\right)^3 + 2(1+3\nu)P_4\left(\tfrac{a}{r}\right)^5\Big\}$$

$$- \cos 6\theta\Big\{6(1+\nu)P_6\left(\tfrac{a}{r}\right)^5 + 4(1+5\nu)P_6\left(\tfrac{a}{r}\right)^7\Big\} + F(\theta)$$

$$\frac{1}{E}\left(\widehat{\theta\theta} - \nu\,\widehat{rr}\right) = \frac{a}{6\mu}\Big[(1-\nu)\frac{Q_0}{2} + 4(3-\nu)\frac{a^2}{r^2}\left(1-f\frac{a^2}{r^2}\right)\left(\frac{a}{r}\right)^2$$

$$+ \frac{4}{3}(5-\nu)\left(2f\frac{a^2}{r^2}-1\right)\left(\frac{a}{r}\right)^4 - \frac{2}{3}(7-\nu)f\left(\frac{a}{r}\right)^6\Big\}$$

$$+ \cos 2\theta\Big\{(1+2\nu)P_2 + 2P_2\left(\tfrac{a}{r}\right)^2 - \left(5-\tfrac{1}{3}\nu\right)\left(\tfrac{a}{r}\right)^4\Big\}$$

$$+ \cos 4\theta\Big\{12(1+\nu)P_4\left(\tfrac{a}{r}\right)^2 + 10(3+\nu)P_4\left(\tfrac{a}{r}\right)^4\Big\}$$

$$+ \cos 6\theta\Big\{30(1+\nu)P_6\left(\tfrac{a}{r}\right)^4 + 24(2+\nu)P_6\left(\tfrac{a}{r}\right)^6\Big\}$$

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$$\frac{1}{2}\left\{\left(\frac{\partial w}{\partial R}\right)^2-\left(\frac{\partial w}{\partial R}\right)^2\right\}+\frac{\partial u}{\partial R}=\frac{1}{E}\left(\widehat{rr}-\nu\widehat{\theta\theta}\right)$$

here

$$\frac{1}{E}\left(\widehat{rr}-\nu\widehat{\theta\theta}\right)=\frac{f}{64}\left[\left\{(1-\nu)\frac{a_0}{2}+4(1-3\nu)\frac{a^2}{R^2}\ldots\right.\right.$$

$$\left.+\frac{4}{3}(1-5\nu)\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^4-\frac{2}{3}(1-7\nu)\ldots\left(\frac{a}{R}\right)^6\right\}$$

$$-\cos 2\theta\left\{(2+\nu)P_2+12\nu P_2\left(\frac{a}{R}\right)^2-\left(\frac{1}{3}-5\nu\right)\left(\frac{a}{R}\right)^4\right\}$$

$$-\cos 4\theta\left\{12(1+\nu)P_4\left(\frac{a}{R}\right)^2+10(1+3\nu)P_4\left(\frac{a}{R}\right)^4\right\}$$

$$-\cos 6\theta\left\{30(1+\nu)P_6\left(\frac{a}{R}\right)^4+28(1+3\nu)P_6\left(\frac{a}{R}\right)^6\right\}$$

$$\frac{1}{2}\left\{\left(\frac{\partial w}{\partial R}\right)^2-\left(\frac{\partial w_0}{\partial R}\right)^2\right\}=\frac{1}{2}\left[\left\{\left(\frac{a}{R}\right)\sin 2\theta-4f\left(\frac{a^2}{R^2}-\frac{a^2}{R^2}\right)\frac{a}{R}\right\}^2-\left\{\frac{a}{R}\sin 2\theta\right\}^2\right]$$

$$=\frac{1}{2}\left(\frac{a}{R}\right)^2 4f\left(\frac{a^2}{R^2}-\frac{a^2}{R^2}\right)\left[4f\left(\frac{a^2}{R^2}-\frac{a^2}{R^2}\right)-2\sin^2\theta\right]$$ * $4f = f$

$$=\frac{f}{2}\left(\frac{a^2}{R^2}-\frac{a^2}{R^2}\right)\frac{a^2}{R^2}\left[f\left(\frac{a^2}{R^2}-\frac{a^2}{R^2}\right)+\cos 2\theta-1\right]$$

$$=\frac{f}{64}\left[\left\{32\frac{a^2}{R^2}\left(f\frac{a^2}{R^2}-1\right)\frac{a^2}{R^2}-32\left(2f\frac{a^2}{R^2}-1\right)\frac{a^4}{R^4}+32f\left(\frac{a}{R}\right)^6\right\}\right.$$

$$\left.+\cos 2\theta\left\{32\frac{a^2}{R^2}\frac{a^2}{R^2}-32\frac{a^4}{R^4}\right\}\right]$$

$$\frac{\partial u}{\partial r} = \frac{p}{64}\Big[\Big\{(1-\nu)\frac{a_1}{2} + 12(3-\nu)\frac{a^2}{R^2}\Big(1 - f\frac{a^2}{R^2}\Big)\Big(\frac{r}{R}\Big)^2$$

$$+ \frac{20}{3}(5-\nu)\Big(2f\frac{a^2}{R^2} - 1\Big)\Big(\frac{r}{R}\Big)^4 - \frac{14}{3}(7-\nu)f\Big(\frac{r}{R}\Big)^6\Big\}$$

$$- \cos 2\theta\Big\{(2+\nu)P_2 + 4\Big(8\frac{a^2}{R^2} + 3\nu P_2\Big)\Big(\frac{r}{R}\Big)^2 - \Big(\frac{44}{3} - 5\nu\Big)\Big(\frac{r}{R}\Big)^4\Big\}$$

$$- \cos 4\theta\Big\{12(1+\nu)P_4\Big(\frac{r}{R}\Big)^2 + 10(1+3\nu)R_4\Big(\frac{r}{R}\Big)^4\Big\}$$

$$- \cos 6\theta\Big\{30(1+\nu)P_6\Big(\frac{r}{R}\Big)^4 + 28(1+5\nu)R_6\Big(\frac{r}{R}\Big)^6\Big\}$$

$$\frac{u}{R} = \frac{p}{64}\Big[\Big\{(1-\nu)\frac{a_1}{2}\Big(\frac{r}{R}\Big) + 4(3-\nu)\frac{a^2}{R^2}\Big(1 - f\frac{a^2}{R^2}\Big)\Big(\frac{r}{R}\Big)^3 + \frac{4}{3}(5-\nu)\Big(2f\frac{a^2}{R^2} - 1\Big)\Big(\frac{r}{R}\Big)^5$$

$$- \frac{2}{3}(7-\nu)f\Big(\frac{r}{R}\Big)^7\Big\}$$

$$- \cos 2\theta\Big\{(2+\nu)P_2\Big(\frac{r}{R}\Big) + \frac{4}{3}\,[illegible]\Big(\frac{r}{R}\Big)^3 - [illegible]\Big\}$$

$$- \cos 4\theta\Big\{4(1+\nu)P_4\Big(\frac{r}{R}\Big)^3 + 2(1+3\nu)R_4\Big(\frac{r}{R}\Big)^5\Big\}$$

$$- \cos 6\theta\Big\{6(1+\nu)P_6\Big(\frac{r}{R}\Big)^5 + 4(1+5\nu)R_6\Big(\frac{r}{R}\Big)^7\Big\}\Big] + F(\theta)$$

$$\therefore \frac{v}{R} = \frac{p}{64}\Big[\cos 2\theta\Big\{\tfrac{3}{2}(1+\nu)P_2\left(\tfrac{r}{R}\right) + 2\left(\tfrac{4}{3}\tfrac{r^2}{R^2} + \overline{3+\nu}\,Q_2\right)\left(\tfrac{r}{R}\right)^3 - \left(\tfrac{86}{15} - \tfrac{2}{3}\nu\right)\left(\tfrac{r}{R}\right)^5\Big\}$$

$$+ \sin 4\theta\Big\{4(1+\nu)P_4\left(\tfrac{r}{R}\right)^3 + 2(4+2\nu)R_4\left(\tfrac{r}{R}\right)^5\Big\}$$

$$+ \sin 6\theta\Big\{6(1+\nu)P_6\left(\tfrac{r}{R}\right)^5 + 2(5+3\nu)R_6\left(\tfrac{r}{R}\right)^7\Big\}\Big] - \int F(\theta)\,d\theta + G\left(\tfrac{r}{R}\right)$$

$$\frac{\int F(\theta)d\theta}{\left(\frac{r}{R}\right)} + \frac{F'(\theta)}{\left(\frac{r}{R}\right)} + G'\left(\tfrac{r}{R}\right) - \frac{G\left(\frac{r}{R}\right)}{\left(\frac{r}{R}\right)} = 0$$

or $\int F(\theta)d\theta + F'(\theta) = G\left(\tfrac{r}{R}\right) - \tfrac{r}{R}\,G'\left(\tfrac{r}{R}\right)$

or $\int F(\theta)d\theta + F'(\theta) = C$ $\qquad \therefore F, G = 0$

$G\left(\tfrac{r}{R}\right) - \tfrac{r}{R}\,G'\left(\tfrac{r}{R}\right) = C$

or $F(\theta) + F''(\theta) = 0$

$F'' + F = 0$ only $F = A \begin{smallmatrix}\sin\theta\\ \cos\theta\end{smallmatrix} >$ cut

$$G'\left(\tfrac{r}{R}\right) - \frac{1}{\left(\frac{r}{R}\right)}G\left(\tfrac{r}{R}\right) = -\frac{C}{\left(\frac{r}{R}\right)}$$

$$\left(\tfrac{r}{R}\right)\frac{d}{d\left(\frac{r}{R}\right)}\Big[\frac{1}{\left(\frac{r}{R}\right)}G\left(\tfrac{r}{R}\right)\Big] = -\frac{C}{\left(\frac{r}{R}\right)} \quad \Big\| \quad \frac{1}{\left(\frac{r}{R}\right)}G = \frac{C}{\left(\frac{r}{R}\right)} + B$$

$$G = C + B\left(\tfrac{r}{R}\right)$$

the undisturbed stress function outside the circular region

$$\frac{\phi_1}{R^2} = \frac{1}{4}\sigma(1-\cos 2\theta)\left(\frac{a}{R}\right)^2$$

[illegible]

$$\frac{\phi_2}{R^2} = \sigma\Big[P_0 \log\frac{a}{R} + \cos 2\theta\left\{Q_2\frac{1}{(\frac{a}{R})^2} + S_2\right\}$$
$$+ \cos 4\theta\left\{Q_4\frac{1}{(\frac{a}{R})^4} + S_4\frac{1}{(\frac{a}{R})^2}\right\}$$
$$+ \cos 6\theta\left\{Q_6\frac{1}{(\frac{a}{R})^6} + S_6\frac{1}{(\frac{a}{R})^4}\right\}\Big]$$

$$\frac{1}{\frac{a}{R}}\frac{\partial(\frac{\phi_2}{R^2})}{\partial\frac{a}{R}} = \sigma\Big[P_0\frac{1}{(\frac{a}{R})^2} + \cos 2\theta\left\{-\frac{2Q_2}{(\frac{a}{R})^4}\right\}$$
$$+ \cos 4\theta\left\{-\frac{4Q_4}{(\frac{a}{R})^6} - \frac{2S_4}{(\frac{a}{R})^4}\right\}$$
$$+ \cos 6\theta\left\{-\frac{6Q_6}{(\frac{a}{R})^8} - \frac{4S_6}{(\frac{a}{R})^6}\right\}\Big]$$

$$\frac{1}{(\frac{a}{R})^2}\frac{\partial^2(\frac{\phi_2}{R^2})}{\partial\theta^2} = \sigma\Big[-4\cos 2\theta\left\{\frac{Q_2}{(\frac{a}{R})^4} + \frac{S_2}{(\frac{a}{R})^2}\right\} - 16\cos 4\theta\left\{\frac{Q_4}{(\frac{a}{R})^6} + \frac{S_4}{(\frac{a}{R})^4}\right\}$$
$$-36\cos 6\theta\left\{\frac{Q_6}{(\frac{a}{R})^8} + \frac{S_6}{(\frac{a}{R})^6}\right\}\Big]$$

$$\widehat{rr}_1 = \sigma\left[\frac{R_0}{\left(\frac{r}{R}\right)^2} - \cos 2\theta \left\{ \frac{6Q_2}{\left(\frac{r}{R}\right)^4} + \frac{4S_2}{\left(\frac{r}{R}\right)^2} \right\}\right.$$

$$\left. - \cos 4\theta \left\{ \frac{20\,Q_4}{\left(\frac{r}{R}\right)^6} + \frac{18\,S_4}{\left(\frac{r}{R}\right)^4} \right\} - \cos 6\theta \left\{ \frac{42\,Q_6}{\left(\frac{r}{R}\right)^8} + \frac{40\,S_6}{\left(\frac{r}{R}\right)^6} \right\} \right]$$

$$\widehat{\theta\theta}_1 = \sigma\left[- \frac{R_0}{\left(\frac{r}{R}\right)^2} + \cos 2\theta \frac{6Q_2}{\left(\frac{r}{R}\right)^4} + \cos 4\theta \left\{ \frac{20\,Q_4}{\left(\frac{r}{R}\right)^6} + \frac{6\,S_4}{\left(\frac{r}{R}\right)^4} \right\}\right.$$

$$\left. + \cos 6\theta \left\{ \frac{42\,Q_6}{\left(\frac{r}{R}\right)^8} + \frac{30\,S_6}{\left(\frac{r}{R}\right)^6} \right\} \right]$$

$$\widehat{r\theta}_1 = \sigma\left[-2\sin 2\theta \left\{ \frac{3Q_2}{\left(\frac{r}{R}\right)^4} + \frac{S_2}{\left(\frac{r}{R}\right)^2} \right\} - 4\sin 4\theta \left\{ \frac{5Q_4}{\left(\frac{r}{R}\right)^6} + \frac{3\,S_4}{\left(\frac{r}{R}\right)^4} \right\}\right.$$

$$\left. - 6\sin 6\theta \left\{ \frac{7Q_6}{\left(\frac{r}{R}\right)^8} + \frac{5\,S_6}{\left(\frac{r}{R}\right)^6} \right\} \right]$$

Therefore
$$\widehat{rr} = \sigma\left[\frac{1}{2} + \frac{R_0}{\left(\frac{r}{R}\right)^2} + \cos 2\theta \left\{ \frac{1}{2} - \frac{6Q_2}{\left(\frac{r}{R}\right)^4} - \frac{4S_2}{\left(\frac{r}{R}\right)^2} \right\}\right.$$

$$\left. - \cancel{\cos 4\theta \left\{ \frac{20\,Q_4}{\left(\frac{r}{R}\right)^6} + \frac{18\,S_4}{\left(\frac{r}{R}\right)^4} \right\} - \cos 6\theta \left\{ \frac{42\,Q_6}{\left(\frac{r}{R}\right)^8} + \frac{40\,S_6}{\left(\frac{r}{R}\right)^6} \right\}} \right]$$

$$\widehat{\theta\theta} = \sigma\left[\frac{1}{2} - \frac{R_0}{\left(\frac{r}{R}\right)^2} + \cos 2\theta \left\{ \frac{6Q_2}{\left(\frac{r}{R}\right)^4} - \frac{1}{2} \right\} + \cancel{\cos 4\theta \left\{ \frac{20\,Q_4}{\left(\frac{r}{R}\right)^6} + \frac{6\,S_4}{\left(\frac{r}{R}\right)^4} \right\}}\right.$$

$$\left. + \cos 6\theta \left\{ \frac{42\,Q_6}{\left(\frac{r}{R}\right)^8} + \frac{30\,S_6}{\left(\frac{r}{R}\right)^6} \right\} \right]$$

$$\widehat{r\theta} = -\sigma\left[\sin 2\theta\left\{\frac{1}{2} + \frac{6Q_2}{(\frac{a}{R})^4} + \frac{2S_2}{(\frac{a}{R})^2}\right\} + \sin 4\theta\left\{\frac{20Q_4}{(\frac{a}{R})^6} + \frac{12S_4}{(\frac{a}{R})^4}\right\}\right.$$
$$\left. + \sin 6\theta\left\{\frac{42Q_6}{(\frac{a}{R})^8} + \frac{30S_6}{(\frac{a}{R})^6}\right\}\right]$$

The stress conditions at the boundary of the circular region are then

$$\sigma\left\{\frac{1}{2} + \frac{R_0}{(\frac{a}{R})^2}\right\} = \frac{Ef}{64}\left\{\frac{1}{2}Q_0 + 4\left(\frac{a}{R}\right)^2\left(1 - f\frac{a^2}{R^2}\right) + \frac{4}{3}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{a}{R}\right)^4 - \frac{2}{3}f\left(\frac{a}{R}\right)^6\right\} \quad (1)$$

$$\sigma\left\{\frac{6Q_2}{(\frac{a}{R})^4} + \frac{4S_2}{(\frac{a}{R})^2} - \frac{1}{2}\right\} = \frac{Ef}{64}\left\{2P_2 + \frac{1}{3}\left(\frac{a}{R}\right)^4\right\} \quad (2)$$

$$\sigma\left\{\frac{20Q_4}{(\frac{a}{R})^6} + \frac{12S_4}{(\frac{a}{R})^4}\right\} = \frac{Ef}{64}\left\{12P_4\left(\frac{a}{R}\right)^2 + 10\,[illegible]\right\} \quad (3)$$

$$\sigma\left\{\frac{42Q_6}{(\frac{a}{R})^8} + \frac{40S_6}{(\frac{a}{R})^6}\right\} = \frac{Ef}{64}\left\{30P_6\left(\frac{a}{R}\right)^4 + 28R_6\left(\frac{a}{R}\right)^6\right\} \quad (4)$$

$$\sigma\left\{\frac{1}{2} - \frac{R_0}{(\frac{a}{R})^2}\right\} = \frac{Ef}{64}\left\{\frac{1}{2}Q_0 + 12\left(\frac{a}{R}\right)^2\left(1 - f\frac{a^2}{R^2}\right) + \frac{20}{3}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{a}{R}\right)^4 - \frac{5}{3}f\left(\frac{a}{R}\right)^6\right\} \quad (5)$$

$$\sigma\left\{\frac{6Q_2}{(\frac{a}{R})^4} - \frac{1}{2}\right\} = \frac{Ef}{64}\left\{P_2 + 12R_2\left(\frac{a}{R}\right)^2 - 5\left(\frac{a}{R}\right)^4\right\} \quad (6)$$

$$\sigma\left\{\frac{20Q_4}{(\frac{a}{R})^6} + \frac{6S_4}{(\frac{a}{R})^4}\right\} = \frac{Ef}{64}\left\{12P_4\left(\frac{a}{R}\right)^2 + 30R_4\left(\frac{a}{R}\right)^4\right\} \quad (7)$$

$$\sigma\left\{\frac{42Q_6}{(\frac{a}{R})^8} + \frac{90S_6}{(\frac{a}{R})^6}\right\} = \frac{Ef}{64}\left\{30P_6\left(\frac{a}{R}\right)^4 + 56R_6\left(\frac{a}{R}\right)^6\right\} \quad (8)$$

$$-\sigma\left\{\frac{1}{2}+\frac{6Q_2}{(\frac{a}{R})^4}+\frac{8Q_2}{(\frac{a}{R})^2}\right\}=\frac{E\delta}{R}\left\{[illegible]+6Q_2\left(\frac{a}{R}\right)^2-\frac{5}{3}\left(\frac{a}{R}\right)^4\right\} \quad (9)$$

$$-\sigma\left\{\frac{[illegible]Q_4}{(\frac{a}{R})^6}+\frac{[illegible]Q_4}{(\frac{a}{R})^4}\right\}=\frac{E\delta}{6R}\left\{[illegible]\left(\frac{a}{R}\right)^{[illegible]}+20\,Q_4\left(\frac{a}{R}\right)^2\right\} \quad (10)$$

$$-\sigma\left\{\frac{42\,S_6}{(\frac{a}{R})^8}+\frac{30\,Q_6}{(\frac{a}{R})^6}\right\}=\frac{E\delta}{6R}\left\{30\,P_6\left(\frac{a}{R}\right)^4+42\,S_6\left(\frac{a}{R}\right)^2\right\} \quad (11)$$

$$\frac{u}{R}=\frac{\sigma}{E}\Big[\tfrac{1}{2}(1-\nu)\left(\tfrac{a}{R}\right)-(1+\nu)P_0\frac{1}{(\frac{a}{R})}+\cos 2\theta\Big\{[illegible]+2Q_2(1+\nu)\frac{1}{(\frac{a}{R})^{[illegible]}}+\frac{4Q_2}{[illegible]}\Big\}$$

$$+\cos 4\theta\Big\{[illegible]+2(3-\nu)\frac{Q_4}{(\frac{a}{R})^3}\Big\}-[illegible]\Big\{6(1+\nu)\frac{Q_6}{[illegible]}+[illegible]\frac{S_6}{(\frac{a}{R})^{[illegible]}}\Big\}\Big]$$

$$\frac{1}{E}\left(\sigma_\theta-\nu\sigma_r\right)=\frac{\sigma}{E}\Big[\tfrac{1}{2}(1-\nu)-(1+\nu)P_0\frac{1}{(\frac{a}{R})^2}+\cos 2\theta\Big\{[illegible]+\frac{6(1+\nu)Q_2}{(\frac{a}{R})^4}+\frac{4Q_2}{(\frac{a}{R})^2}\Big\}$$

$$+\cos 4\theta\Big\{\frac{30(1+\nu)Q_4}{(\frac{a}{R})^6}+\frac{6(1+3\nu)S_4}{(\frac{a}{R})^4}\Big\}+\cos 6\theta\Big\{\frac{42(1+\nu)Q_6}{(\frac{a}{R})^8}+\frac{30(1+\nu)S_6}{(\frac{a}{R})^6}\Big\}\Big]$$

$$\frac{v}{R}=\frac{\sigma}{E}\Big[\sin 2\theta\Big\{\frac{2(1+\nu)Q_2}{(\frac{a}{R})^3}\qquad-\tfrac{1}{2}(1+\nu)\left(\tfrac{a}{R}\right)\Big\}$$

$$+\sin 4\theta\Big\{\frac{4(1+\nu)Q_4}{(\frac{a}{R})^5}+\frac{4\nu\,S_4}{(\frac{a}{R})^3}\Big\}+\sin 6\theta\Big\{\frac{6(1+\nu)Q_6}{(\frac{a}{R})^7}+\frac{(2+6\nu)S_6}{(\frac{a}{R})^5}\Big\}\Big]$$

[illegible] the [illegible] we [illegible] the [illegible] the [illegible] region is 308

$$* \quad [illegible] - (1+\nu)\frac{P_0}{(\frac{a}{R})^2} [illegible] = \frac{Et}{64}\Big\{(1+\nu)\frac{Q_0}{2} + 4(3+\nu)\frac{a}{R}(1-2\frac{a^2}{R^2}) + \frac{4}{3} [illegible] - \frac{2}{3}(7+\nu) [illegible] \Big\}$$

$$-\sigma\Big\{\frac{1+\nu}{2} + \frac{2Q_2 [illegible]}{(\frac{a}{R})^4} + \frac{[illegible] S_2}{(\frac{a}{R})^2}\Big\} = \frac{Et}{64}\Big\{ [illegible] \Big\}$$

$$-\sigma\Big\{\frac{4(1+\nu)S_4}{(\frac{a}{R})^6} + \frac{2(3+\nu)S_4}{(\frac{a}{R})^4}\Big\} = \frac{Et}{64}\Big\{4(1+\nu)R_4(\frac{a}{R})^2 + 2(1+3\nu)R_4(\frac{a}{R})^4\Big\}$$

$$-\sigma\Big\{\frac{6(1+\nu)S_6}{(\frac{a}{R})^8} + \frac{4(2+\nu)S_6}{(\frac{a}{R})^6}\Big\} = \frac{Et}{64}\Big\{6(1+\nu)R_6(\frac{a}{R})^4 + 2(5+3\nu)R_6(\frac{a}{R})^6\Big\}$$

$$\sigma\Big\{\frac{2(1+\nu)S_2}{(\frac{a}{R})^4} - \frac{1}{2}(1+\nu)\Big\} = \frac{Et}{64}\Big\{\frac{3}{2}(1+\nu) [illegible] + 2 [illegible] + 3(1+\nu)R_2(\frac{a}{R})^2 [illegible] \Big\} \quad (16)$$

$$\sigma\Big\{\frac{4(1+\nu)S_4}{(\frac{a}{R})^6} + \frac{4\nu S_4}{(\frac{a}{R})^4}\Big\} = \frac{Et}{64}\Big\{4(1+\nu)R_4(\frac{a}{R})^2 + 2(1+3\nu)R_4(\frac{a}{R})^4\Big\} \quad (17)$$

$$\sigma\Big\{\frac{6(1+\nu)S_6}{(\frac{a}{R})^8} + \frac{2(1+3\nu)S_6}{(\frac{a}{R})^6}\Big\} = \frac{Et}{64}\Big\{6(1+\nu)R_6(\frac{a}{R})^4 + 2(5+3\nu)R_6(\frac{a}{R})^6\Big\} \quad (18)$$

Let us investigate the equations (3), (6), (9), (13), & (16)

$$6q_2 + 4s_2 - \frac{1}{2} = \left\{ 3p_2' + \frac{1}{3}h \right\} \qquad h = \left(\frac{a}{R}\right)^2$$

$$6q_2 - \frac{1}{2} = -\left\{ p_2' + 12r_2 - 5h \right\}$$

$$-\left(6q_2 + 2s_2 + \frac{1}{2}\right) = \left\{ 3p_2' + 6r_2' - \frac{5}{3}h' \right\}$$

$$\left[\frac{1+\nu}{2} + 2(1+\nu)q_2 + 4s_2\right] = \left\{ (2+\nu)p_2 + 4\nu r_2' + \left(\frac{21}{5}+\nu\right)h \right\}$$

$$\left[2(1+\nu)q_2 - \frac{1}{2}(1+\nu)\right] = \left\{ \frac{3}{2}(1+\nu)p_2' + 2(3+\nu)r_2' - \left(\frac{2}{5} - \frac{2}{3}\nu\right)h' \right\}$$

The question is whether these system of equations are consistent. They are not consistent, so we can only satisfy them approximately, by means of method of least square, thus

$$6\left(6q_2 + 4s_2 - \frac{1}{2} - 3p_2 - \frac{1}{3}h\right) + 6\left(6q_2 - \frac{1}{2} - p_2 - 12r_2 + 5h\right)$$

$$+ 6\left(6q_2 + 2s_2 + \frac{1}{2} + 3p_2 + 6r_2 - \frac{5}{3}h\right)$$

$$+ 2(1+\nu)\left[\frac{1+\nu}{2} + 2(1+\nu)q_2 + 4s_2 + (2+\nu)p_2 + 4\nu r_2 + \left(\frac{21}{5}+\nu\right)h\right]$$

$$+ 2(1+\nu)\left[2(1+\nu)q_2 - \frac{1}{2}(1+\nu) - \frac{3}{2}(1+\nu)p_2 - 2(3+\nu)r_2 + \left(\frac{2}{5} - \frac{2}{3}\nu\right)h\right] = 0$$

or

$$\left[108 + 8(1+\nu)^2\right]q_2 + \left[36 + 8(1+\nu)\right]s_2 + \left[(1-\nu^2) - 6\right]p_2 + \left[4(1+\nu)(-3+\nu) - 36\right]r_2 + \left[-3 + 18h + 2(1+\nu)\left(\frac{23}{5} + \frac{\nu}{3}\right)h\right] = 0 \qquad (A)$$

$$4\left(6q_{12}+4s_2-\frac{1}{2}-9p_2-\frac{h}{3}\right)+2\left(6q_{32}+9s_2+\frac{1}{2}+9p_2+6n_2-\frac{5}{3}h\right)$$

$$+4\left[\frac{1+\nu}{2}+2(1+\nu)q_{12}+4s_2+(9+\nu)p_2+12n_2+\left(\frac{21}{5}+\nu\right)h\right]=0$$

$$\boxed{[36+8(1+\nu)]q_{12}+36s_2+4(1+\nu)p_2+4(3+4\nu)n_2+\left[1+2\nu+\left(\frac{112}{15}+4\nu\right)h\right]=0}$$

(B)

$$2\left(2p_2+\frac{h}{3}-6q_{12}-4s_2+\frac{1}{2}\right)+\left(p_2+12s_2-5h-6q_{32}+\frac{1}{2}\right)$$

$$+2\left(2p_2+6n_2-\frac{5}{3}h+6q_{32}+3s_2+\frac{1}{2}\right)+(3+\nu)\left[(2+\nu)p_2+4\nu n_2+\left(\frac{21}{5}+\nu\right)h\right.$$

$$\left.+\frac{1+\nu}{2}+2(1+\nu)q_{32}+4s_2\right]+\frac{3}{2}(1+\nu)\left[\frac{3}{2}(1+\nu)p_2+2(3+\nu)n_2-\left(\frac{2}{5}\cdot\frac{2}{3}\nu\right)h\right.$$

$$\left.-3(1+\nu)q_{12}+\frac{1}{2}(1+\nu)\right]=0$$

$$[-6+(1-\nu^2)]q_{32}+4(1+\nu)s_2+\left[9+(2+\nu)^2+\frac{9}{4}(1+\nu)^2\right]p_2+\left[24+4(2+\nu)\nu+3(1+\nu)(3+\nu)\right]n_2$$

$$+\left[\frac{5}{2}+\frac{1}{2}(1+\nu)\left(\frac{7}{2}+\frac{5}{2}\nu\right)+h\left\{\frac{2}{3}-5-\frac{10}{3}+(2+\nu)\left(\frac{21}{5}+\nu\right)-\frac{3}{2}(1+\nu)\left(\frac{2}{5}-\frac{2}{3}\nu\right)\right\}\right]=0$$

(C)

$$6\left(p_2 + 12 r_2 - 3h - 5 q_2 + \tfrac{1}{2}\right) + 3\left(2p_2 + 6r_2 - \tfrac{5}{3}h + 6q_2 + 9s_2 + \tfrac{1}{2}\right)$$

$$+ 2\nu\left[(3+\nu)p_2 + 4\nu r_2 + \left(\tfrac{21}{5}+\nu\right)h + \tfrac{1+\nu}{2} + 2(1+\nu)q_2 + 4s_2\right]$$

$$+ (3+\nu)\left[\tfrac{3}{2}(1+\nu)p_2 + 2(3+\nu)r_2 - \left(\tfrac{2}{5} - \tfrac{2}{3}\nu\right)h - 2(1+\nu)q_2 + \tfrac{1}{2}(1+\nu)\right] = 0$$

$$\left[-18 + 4\nu(1+\nu) - 2(1+\nu)(3+\nu)\right]q_2 + \left[6 + 8\nu\right]s_2$$

$$+ \left[12 + 2\nu(3+\nu) + \tfrac{3}{2}(1+\nu)(3+\nu)\right]p_2 + \left[90 + 8\nu^2 + 2(3+\nu)^2\right]r_2$$

$$+ \left[\tfrac{9}{2} + \nu(1+\nu) + \tfrac{1}{2}(1+\nu)(3+\nu) + h\left\{-35 + 2\nu\left(\tfrac{21}{5}+\nu\right) - 2(3+\nu)\left(\tfrac{1}{5} - \tfrac{1}{3}\nu\right)\right\}\right] = 0 \qquad (D)$$

the equations (A), (B), (C), (D) determine $\boxed{p_2, q_2, r_2, s_2}$

$$\widehat{rr} - \widehat{\theta\theta} = \sigma\left[1 + [illegible] - \frac{4S_2}{(a/R)^2}\right]$$

$$\int_0^{2\pi} d\theta\left\{(\widehat{rr} + \widehat{\theta\theta})^2 - 2(1+\nu)(\widehat{rr}\,\widehat{\theta\theta} - \widehat{r\theta}^2)\right\}$$

$$= \pi\sigma^2\left[2 + \frac{16S_2^2}{(a/R)^4} - 2(1+\nu)\left\{\frac{1}{2} - \frac{2R_0^2}{(a/R)^4} - \left(\frac{1}{2} - \frac{6Q_2}{(a/R)^4}\right)^2 + \frac{4S_2}{(a/R)^2}\left(\frac{1}{2} - \frac{6Q_2}{(a/R)^4}\right)\right.\right.$$

$$\left.\left. - \left(\frac{1}{2} + \frac{6Q_2}{(a/R)^4} + \frac{2S_2}{(a/R)^2}\right)^2\right\}\right]$$

$$= \pi\sigma^2\left[2 + \frac{16S_2^2}{(a/R)^4} - 2(1+\nu)\left\{(-2R_0^2 - 4S_2^2)\frac{1}{(a/R)^4} + (-24S_2Q_2 - 24S_2Q_2)\frac{1}{(a/R)^6}\right.\right.$$

$$\left.\left. + (-72Q_2^2)\frac{1}{(a/R)^8}\right\}\right]$$

$\mathcal{E}_1$ = Strain energy outside the circular region — the same at a given [illegible].

$$= \frac{t\sigma^2}{2E}\pi\int_a^\infty r\,dr\left[\frac{16S_2^2}{(a/R)^4} + 2(1+\nu)\left\{\frac{2(R_0^2 + 2S_2^2)}{(a/R)^4} + \frac{48S_2Q_2}{(a/R)^6} + \frac{72Q_2^2}{(a/R)^8}\right\}\right]$$

$$= \frac{t\sigma^2}{2E}\pi R^2\left[8\frac{S_2^2}{(a/R)^2} + 2(1+\nu)\left\{\frac{(R_0^2 + 2S_2^2)}{(a/R)^2} + 12\frac{S_2Q_2}{(a/R)^4} + 12\frac{Q_2^2}{(a/R)^6}\right\}\right]$$

For the extensional strain energy in the circular region

$$\widehat{rr}+\widehat{\theta\theta} = \frac{Ef}{64}\Big[\Big\{Q_0 + 16\frac{a^2}{R^2}\Big(1-f\frac{a^2}{R^2}\Big)\Big(\frac{a}{R}\Big)^2 + 8\Big(2f\frac{a^2}{R^2}-1\Big)\Big(\frac{a}{R}\Big)^4 - \frac{16}{3}f\Big(\frac{a}{R}\Big)^6\Big\}$$

$$+\cos 2\theta\Big\{-P_2 + 12P_2\Big(\frac{a}{R}\Big)^2 - \frac{16}{3}\Big(\frac{a}{R}\Big)^4\Big\}$$

$$\int_0^{2\pi} d\theta\Big\{(\widehat{rr}+\widehat{\theta\theta})^2 - 2(1+\nu)\big(\widehat{rr}\,\widehat{\theta\theta} - \widehat{r\theta}^2\big)\Big\}$$

$$= \pi\frac{E^2f^2}{64^2}\Big[2\Big\{Q_0 + 16\frac{a^2}{R^2}\Big(1-f\frac{a^2}{R^2}\Big)\Big(\frac{a}{R}\Big)^2 + 8\Big(2f\frac{a^2}{R^2}-1\Big)\Big(\frac{a}{R}\Big)^4 - \frac{16}{3}f\Big(\frac{a}{R}\Big)^6\Big\}^2$$

$$+\Big\{-P_2 + 12R_2\Big(\frac{a}{R}\Big)^2 - \frac{16}{3}\Big(\frac{a}{R}\Big)^4\Big\}^2$$

$$-2(1+\nu)\Big\{2\Big[\frac{1}{2}Q_0 + 4\frac{a^2}{R^2}\Big(1-f\frac{a^2}{R^2}\Big)\frac{a^2}{R^2} + \frac{4}{3}\Big(2f\frac{a^2}{R^2}-1\Big)\Big(\frac{a}{R}\Big)^4 - \frac{2}{3}f\Big(\frac{a}{R}\Big)^6\Big]$$

$$\cdot\Big[\frac{1}{2}Q_0 + 12\frac{a^2}{R^2}\Big(1-f\frac{a^2}{R^2}\Big)\frac{a^2}{R^2} + \frac{20}{3}\Big(2f\frac{a^2}{R^2}-1\Big)\Big(\frac{a}{R}\Big)^4 - \frac{14}{3}f\Big(\frac{a}{R}\Big)^6\Big]$$

$$-\Big[2P_2 + \frac{1}{3}\Big(\frac{a}{R}\Big)^4\Big]\Big[P_2 + 12R_2\Big(\frac{a}{R}\Big)^2 - 5\Big(\frac{a}{R}\Big)^4\Big] - 4\Big[P_2 + 3P_2\Big(\frac{a}{R}\Big)^2 - \frac{5}{6}\Big(\frac{a}{R}\Big)^4\Big]^2\Big\}\Big]$$

Extensional strain energy in the circular region, E_2

$$= \frac{\pi R^2 t E f^2}{2 \times 64^2}\Bigg[2\Big\{\tfrac{1}{2}Q_0^2\left(\tfrac{a}{R}\right)^2 + 8Q_0\left(\tfrac{a}{R}\right)^6\left(1-f\tfrac{a^2}{R^2}\right) + \tfrac{16^2}{5}\left(\tfrac{a}{R}\right)^{10}\left(1-f\tfrac{a^2}{R^2}\right)^2 + \tfrac{8}{3}Q_0\left(\tfrac{a}{R}\right)^6\left(2f\tfrac{a^2}{R^2}-1\right) - \tfrac{4}{3}Q_0 f\left(\tfrac{a}{R}\right)^8$$

$$+ \tfrac{64}{10}\left(2f\tfrac{a^2}{R^2}-1\right)^2\left(\tfrac{a}{R}\right)^{10} +$$

$$+ 32\left(\tfrac{a}{R}\right)^{10}\left(1-f\tfrac{a^2}{R^2}\right)\left(2f\tfrac{a^2}{R^2}-1\right) - \tfrac{64}{9}f\left(2f\tfrac{a^2}{R^2}-1\right)\left(\tfrac{a}{R}\right)^{12} + \tfrac{8\times16}{21}f^2\left(\tfrac{a}{R}\right)^{14} - \tfrac{32\times16}{3}f\left(1-f\tfrac{a^2}{R^2}\right)\left(\tfrac{a}{R}\right)^{12}\Big\}$$

$$+ \Big\{\tfrac{1}{2}P_2^2\left(\tfrac{a}{R}\right)^2 - 6R_2P_2\left(\tfrac{a}{R}\right)^4 + \tfrac{16}{9}P_2\left(\tfrac{a}{R}\right)^6 + 24R_2^2\left(\tfrac{a}{R}\right)^6 - 16R_2\left(\tfrac{a}{R}\right)^8 + \tfrac{16\times16}{3}\left(\tfrac{a}{R}\right)^{10}\Big\}$$

$$-4(1+\nu)\Big\{\tfrac{1}{4}Q_0^2\left(\tfrac{a}{R}\right)^2 + 2Q_0\left(\tfrac{a}{R}\right)^6\left(1-f\tfrac{a^2}{R^2}\right) + 8\left(\tfrac{a}{R}\right)^{10}\left(1-f\tfrac{a^2}{R^2}\right)^2 + \tfrac{2}{3}Q_0\left(2f\tfrac{a^2}{R^2}-1\right)\left(\tfrac{a}{R}\right)^6 - \tfrac{1}{3}Q_0 f\left(\tfrac{a}{R}\right)^8$$

$$+ \tfrac{16}{3}\left(\tfrac{a}{R}\right)^{10}\left(1-f\tfrac{a^2}{R^2}\right)\left(2f\tfrac{a^2}{R^2}-1\right) - \tfrac{8}{3}f\left(1-f\tfrac{a^2}{R^2}\right)\left(\tfrac{a}{R}\right)^{12} + \tfrac{8}{9}\left(2f\tfrac{a^2}{R^2}-1\right)^2\left(\tfrac{a}{R}\right)^{10} - \tfrac{2}{3}f\left(2f\tfrac{a^2}{R^2}-1\right)\left(\tfrac{a}{R}\right)^{12}$$

$$+ \tfrac{2}{7}f^2\left(\tfrac{a}{R}\right)^{14}\Big\}$$

$$+ 8(1+\nu)\Big\{3P_2^2\left(\tfrac{a}{R}\right)^2 + 12R_2P_2\left(\tfrac{a}{R}\right)^4 - \tfrac{44}{18}P_2\left(\tfrac{a}{R}\right)^6 - \tfrac{3}{4}R_2\left(\tfrac{a}{R}\right)^8 + \tfrac{1}{48}\left(\tfrac{a}{R}\right)^{10} + 9R_2\left(\tfrac{a}{R}\right)^6\Big\}$$

$$+ 8(1+\nu)\Big\{\tfrac{1}{2}P_2^2\left(\tfrac{a}{R}\right)^2 + \tfrac{3}{2}R_2P_2\left(\tfrac{a}{R}\right)^4 + \tfrac{3}{4}R_2^2\left(\tfrac{a}{R}\right)^6 - \tfrac{5}{18}P_2\left(\tfrac{a}{R}\right)^6 - \tfrac{5}{16}R_2\left(\tfrac{a}{R}\right)^8 + \tfrac{25}{360}\left(\tfrac{a}{R}\right)^{10}\Big\}\Bigg]$$

[illegible]

$k_1 = $ [illegible] $= \frac{1}{R}$ [illegible]

$k_2 = \frac{1}{R}$ [illegible]

$= \frac{1}{R}$ [illegible]

$\tau = 0$

$k\tau_2 = \frac{1}{R}$ [illegible]

$k_1 \tau_2 = \frac{1}{R^2}$ [illegible] $- 4(\frac{a}{R})^2$ [illegible] $+ 3$ [illegible]

The bending strain energy [illegible]

$$\frac{1}{24}\frac{t^3}{(1-\nu^2)} E \cdot 2\pi \int_0^{\frac{a}{R}} \Big[4 \text{ [illegible]} \Big]$$

$$= \frac{1}{3}\frac{t^3 E \pi}{(1-\nu^2)} \int_0^{\frac{a}{R}} \Big[2(\tfrac{a}{R})^4 - 8(\tfrac{a}{R})^2(\tfrac{r}{R})^2 + 8(\tfrac{r}{R})^4 - (1+\nu)\{ (\tfrac{a}{R})^4 - 4 \text{ [illegible]} \} \Big]$$

$$= \frac{1}{3}\frac{t^3 E \pi R^2}{(1-\nu^2)} \Big[1 - 2 + \frac{4}{3} - (1+\nu)(\tfrac{1}{2} - 1 + \tfrac{1}{2}) \Big] (\tfrac{a}{R})^6$$

$$= \frac{1}{9}\frac{t^3 E \pi R^2}{(1-\nu^2)} (\tfrac{a}{R})^6 = E_3$$

[illegible]

the decrease in potential of σ [now - case of uniform stress] 311

$$= t R^2 \frac{\sigma^2}{E} \int_0^{2\pi} d\theta \Big[\tfrac{1}{2}(1-\nu)R_0 - \tfrac{1}{2}(1+\nu)R_0 + \cos^2\theta\left(-2(1+\nu)S_2 + 2S_2\right)$$

$$- d\theta\left[-(1+\nu)S_2\right]\sin^2\theta \Big]$$

$$= t\frac{\sigma^2}{E}\pi R^2\left[-2\nu R_0 - 2\nu S_2 + (1+\nu)S_2\right]$$

$$= t\frac{\sigma^2}{E}\pi R^2\left[(1-\nu)S_2 - 2\nu R_0\right] = f(\sigma)$$

In order to simplify the calculation. [Note: diff. from p. 309] !!!

Put $\dfrac{R_0}{\left(\frac{a}{R}\right)^2} = r_0$, $\quad t\dfrac{a^2}{R^2} = g$, $\quad \dfrac{Q_0}{\left(\frac{a}{R}\right)^4} = q_0$

$\dfrac{Q_2}{\left(\frac{a}{R}\right)^4} = q_2$, $\quad \dfrac{S_2}{\left(\frac{a}{R}\right)^2} = s_2$, $\quad \dfrac{P_2}{\left(\frac{a}{R}\right)^4} = p_2$,

$\dfrac{R_2}{\left(\frac{a}{R}\right)^2} = r_2$

Important !!! Change your (A), (B), (C), (D)

With this set of notation [illegible]

$$\frac{\mathfrak{S}_3}{R^3} = \frac{t}{R}\frac{c^2}{3E}\pi\left(\frac{a}{R}\right)^2\left[\mathcal{P}_3^2 + 2 [illegible] \left\{ [illegible] \right\} \right]$$

$$\frac{\mathfrak{S}_3}{R^3} = \left(\frac{t}{R}\right)^3 \frac{8}{9} E\pi c^2 \left(\frac{a}{R}\right)^2 \frac{1}{1-\nu^2}\ \frac{A}{16} \quad [illegible]$$

$$\frac{\mathfrak{D}}{R^3} = \left(\frac{t}{R}\right)\frac{c^2}{E}\pi\left(\frac{a}{R}\right)^2\left\{(1-\nu)s_3 - [illegible]\right\}.$$

$$\frac{\mathfrak{S}_2}{R^3} = \left(\frac{t}{R}\right)\frac{1}{8192} E c^2 \left(\frac{a}{R}\right)^2 \left[\left\{ \frac{1}{\rho}q_0^2 + 16 [illegible] + \frac{256}{3} [illegible] + \frac{16}{3} q_0 [illegible] \right.\right.$$

$$+ 64 [illegible] - \frac{32}{9} [illegible]$$

$$\left. + \frac{256}{81}c^2 - \frac{1024}{9} [illegible] \right\}$$

$$+ \left\{\frac{1}{2}p_2^2 - 6 n_2 p_2 + \frac{11}{9}p_2 + 24 n_2^2 - 16 n_2 + \frac{[illegible]}{3}\right\}$$

$$-(1+\nu)\left\{\frac{1}{2}q_0^2 + 8 q_0(1-q) + 32(1-q)^2 + \frac{4}{3}q_0(2q-1) - \frac{4}{3}q_0 c\right.$$

$$\left. + \frac{64}{3}(1-q)(2q-1) - \frac{32}{3}q(1-q) + \frac{32}{9}(2q-1)^2 - \frac{32}{3}q(2q-1) + \frac{8}{9}c^2\right\}$$

$$\left. + (1+\nu)\left\{6 p_2^2 + 24 p_2 n_2 - \frac{49}{9}p_2 - \frac{3}{2}n_2 + \frac{7}{24} + 16 n_2^2\right\}\right]$$

The equations for determining the values of r_0 + q_0

$$\frac{1}{2} + r_0 = \left[\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2\right] g \left\{ \frac{1}{2} q_0 + 4(1-g) + \frac{4}{3}(2g-1) - \frac{8}{3} g \right\}$$

$$\frac{1}{2} - r_0 = \left[\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2\right] g \left\{ \frac{1}{2} q_0 + 12(1-g) + \frac{20}{3}(2g-1) - \frac{8}{3} g \right\}$$

$$1 = \left[\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2\right] g \left\{ q_0 + 16(1-g) + 8(2g-1) - \frac{16}{3} g \right\}$$

$$q_0 = \frac{1}{g} \frac{64\sigma}{E\left(\frac{a}{R}\right)^2} - 16(1-g) - 8(2g-1) + \frac{16}{3} g$$

$$\boxed{q_0 = \frac{1}{g} \frac{64\sigma}{E\left(\frac{a}{R}\right)^2} - 8 + \frac{16}{3} g}$$

If $\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2 = \xi$

$$\boxed{q_0 = \frac{1}{g\xi} - 8 + \frac{16}{3} g}$$

$$r_0 = \left[\frac{E}{64\sigma} \frac{a^2}{R^2}\right] g \left\{ \frac{1}{2g} \frac{64\sigma}{E\left(\frac{a}{R}\right)^2} - 4 + \frac{8}{3} g + \frac{8}{3} - 2g \right\} - \frac{1}{2}$$

$$\boxed{r_0 = \left[\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2\right] g \left\{ \frac{2}{3} g - \frac{4}{3} \right\}}$$

$$\boxed{r_0 = \xi g \left(\frac{2}{3} g - \frac{4}{3} \right)}$$

$$\frac{\overline{E}_2}{R^2} = \frac{t}{r_1}\frac{E(\frac{r_2}{r_1})}{6.1\,9^2}\Big[\Big\{q_0^2 + \frac{4}{3}(4-3\xi)q_0 + 34.1333 - 49.7728\xi + 26.4127\xi^2\Big\}$$

$$+ \Big\{\frac{4}{3}p_2^2 - [illegible]\,p_2 + \frac{11}{9}p_2 + [illegible]\,r_2^2 - 1.5\,r_2 - \frac{[illegible]}{3}\Big\}$$

$$-(1+\nu)\Big\{\frac{1}{2}q_0^2 + 4(\tfrac{4}{3}-\xi)q_0 + 14.222 - 14[illegible]\,\xi - 6.[illegible]\,\xi^2\Big\}$$

$$+(1+\nu)\Big\{6p_2^2 + 24\,p_2 r_2 - \frac{49}{9}p_2 - \frac{3}{2}r_2 + 18\,r_2^2 + \frac{7}{24}\Big\}\Big]$$

The four equations for the unknowns r_2, s_2, q_2, p_2 are now:

$\nu = 0.3 \qquad 1+\nu = 1.3 \qquad (1+\nu)^2 = 1.69$

$1-\nu^2 = 0.91$

$$114.76\,q_2 + 46.4\,s_2 - 5.0[illegible](\xi p_2) - 50.4\,\xi r_2 + [3.0[illegible]\,\xi - 3] = 0$$

$$46.4\,q_2 + 36\,s_2 + 5.2[illegible](\xi p_2) + 16.85\,\xi r_2 + [13.3333\,\xi + 1.6] = 0$$

$$-5.09\,q_2 + 5.2\,s_2 + 19.565(\xi p_2) + 39.63(\xi r_2) + [2.29333\,\xi + 5.2625] = 0$$

$$-25.02\,q_2 + 8.4\,s_2 + 19.815(\xi p_2) + 122.50(\xi r_2) + [-32.96\,\xi + 7.035] = 0$$

or 1) $q_2 + 0.40432\,s_2 - 0.044353(\xi p_2) - 0.43604(\xi r_2) + [0.26333\,\xi - 0.026142] = 0$

2) $q_2 + 0.77586\,s_2 + 0.11207(\xi p_2) + 0.36307(\xi r_2) + [0.28736\,\xi + 0.034483] = 0$

3) $-q_2 + 1.02161\,s_2 + 3.84332(\xi p_2) + 7.78585(\xi r_2) + [0.45056\,\xi + 1.03387] = 0$

4) $-q_2 + 0.33573\,s_2 + 0.79197(\xi p_2) + 4.89608(\xi r_2) + [-1.31735\,\xi + 0.28118] = 0$

$$1.42593 S_2 + 3.77297(\xi^0_l/r_2) + 7.34981(\xi^0_l r_2) + [0.71389\,\xi^0_l + 1.00725] = 0$$

$$1.79742 S_2 + 3.95539(\xi^0_l/r_2) + 8.14[illegible](\xi^0_l r_2) + [0.73792\,\xi^0_l + 1.06837] = 0$$

$$1.11159 S_2 + 3.10444(\xi^0_l/r_2) + 5.25815(\xi^0_l r_2) + [-1.02799\,\xi^0_l + 0.31566] = 0$$

$$S_2 + 2.64421(\xi^0_l/r_2) + 5.15440(\xi^0_l r_2) + [0.50065\,\xi^0_l + 0.70673] = 0$$

$$S_2 + 2.20053(\xi^0_l/r_2) + 4.53299(\xi^0_l r_2) + [0.41053\,\xi^0_l + 0.57437] = 0$$

$$S_2 + 2.79[illegible]9(\xi^0_l/r_2) + 4.73030(\xi^0_l r_2) + [-0.92459\,\xi^0_l + 0.28397] = 0$$

$$1.45092(\xi^0_l/r_2) + 0.4[illegible]10(\xi^0_l r_2) + [1.42727\,\xi^0_l + 0.42276] = 0$$

$$0.46238(\xi^0_l/r_2) + 0.62141(\xi^0_l r_2) + [0.09012\,\xi^0_l + 0.11231] = 0$$

$$\xi^0_l/r_2 + 0.22913(\xi^0_l r_2) + [0.77110\,\xi^0_l + 0.22541] = 0$$

$$\xi^0_l/r_2 + 1.34017(\xi^0_l r_2) + [0.19436\,\xi^0_l + 0.24232] = 0$$

Thus
$$\xi^0_l r_2 = \frac{0.57674\,\xi^0_l - 0.01391}{1.11107}$$

$$\boxed{\xi^0_l r_2 = 0.51910\,\xi^0_l - 0.01252}$$

$$2(\xi^0_l/r_2) + 1.51930(0.51910\,\xi^0_l - 0.01252) + (0.91546\,\xi^0_l + 0.47073) = 0$$

$$\boxed{\xi^0_l/r_2 = -(0.8[illegible]\,\xi^0_l + 0.2[illegible]54)}$$

$$3C_2 - 1.4423(0.8\ldots\,\xi_c^2 + \ldots) + 1.4412(0.51\ldots\,\xi_c^2 - 0.01252)$$
$$+ [-0.01541\,\xi_c^2 + 1.56037] =$$

$$3C_2 - (\ldots - 1.2\ldots) + (2.44422\,\xi_c^2 - 0.1\ldots)$$
$$+ (-0.01541\,\xi_c^2 + 1.5\ldots) = 0$$

$$\boxed{C_2 = -(0.80505\,\xi_c^2 + 0.04431)}$$

$$2q_2 - 1.14016(0.80505\,\xi_c^2 + 0.04431) - 0.04722(0.9904\,\xi_c^2 + 0.32554)$$
$$- 0.07392(0.518\ldots\,\xi_c^2 - 0.0\ldots) + (0.55069\,\xi_c^2 + 0.\ldots 34) = 0$$

$$2q_2 - (0.95010\,\xi_c^2 + 0.04875) - (0.06027\,\xi_c^2 + 0.01527)$$
$$- (0.03840\,\xi_c^2 - 0.00926) + (0.55069\,\xi_c^2 + 0.10134) = 0$$

$$\boxed{q_2 = 0.24904\,\xi_c^2 + 0.02\ldots 8}$$

We have thus

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2 \frac{\sigma^2}{3E}\left[(0.80505\,\xi g + 0.04131)^2 + 2(1+\nu)\left\{(\xi g)^2(g-2)^2\frac{4}{9} + 2(0.80505\,\xi g + 0.04131)\right\}\right.$$

$$\left.-12(0.80505\,\xi g + 0.04131)(0.24904\,\xi g + 0.02736) + 12(0.24904\,\xi g + 0.02736)^2\right\}\Big]$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2 \frac{\sigma^2}{3E}\Big[62(0.80505\,\xi g + 0.04131)^2 + 1.15556(\xi g)^2(g^2 - 4g + 4)$$

$$- 31.20(0.24904\,\xi g + 0.02736)(0.55191\,\xi g + 0.01393)\Big]$$

$$\boxed{\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2 \frac{\sigma^2}{3E}\Big[(\xi g)^2(1.15556\,g^2 - 4.62222\,g + 43.1326) - 0.17159(\xi g) - 0.001320\Big]}$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a^2}{R^2}\right)^2 \frac{\sigma^2}{3E}\Big[(\xi g)^2(34.5016\,g^2 - 31.2690\,g + 24.5569)$$

$$+(3.73333 - 3.8g)(\xi g)\left[1 - 8\left(1 - \tfrac{2}{3}g\right)(\xi g)\right] + 53\,\xi g(0.69004\,\xi g + 0.22554)$$

$$- 1795(\xi g)(0.51905\,\xi g - 0.01252) + 63(0.69004\,\xi g + 0.22554)^2$$

$$- 2520(0.69004\,\xi g + 0.22554)(0.5190\,\xi g - 0.01252) + 424(0.51905\,\xi g - 0.01252)^2\Big]$$

$$\boxed{\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\pi\frac{a^2}{R^2}\frac{\sigma^2}{3E}\Big[(\xi g)^2(19.5613\,g^2 + 11.0221\,g - 3.2066) + \xi g(5.20071 - 2.8g) + 0.50079\Big]}$$

$$\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right)^3 \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\left[\frac{4096}{9(1-\nu^2)} \quad \frac{1}{\left(\frac{a}{R}\right)^4}(\xi g)^2\right]$$

$$\frac{\mathcal{E}_3}{R^3} = \frac{t}{R}^3 \pi \frac{a}{R}^2 \frac{\sigma^2}{2E}\left[\frac{1}{9(1-\nu^2)} \quad \frac{g^2}{\left(\frac{\sigma}{E}\right)^2}\right]$$

$$\boxed{\frac{\mathcal{E}_3}{R^3} = \frac{t}{R}\pi \frac{a}{R}^2 \frac{\sigma^2}{2E}\left[0.122100 \quad \frac{g^2}{\left(\frac{\sigma}{E}\frac{R}{t}\right)^2}\right]}$$

$$\frac{\mathcal{P}}{R^3} = \frac{t}{R}\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{2E}\left[-[illegible] + (0.20[illegible]5\,\xi g + 0.[illegible]31) - 1.2\,\xi g\left(\tfrac{1}{3}g - \tfrac{5}{3}\right)\right]$$

$$\boxed{-\frac{\mathcal{P}}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{2E}\left[\xi g(0.8g - 0.4743) + 0.057434\right]}$$

Total potential of the system.

$$\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{2E}\Big[(\xi g)^2(20.7259\,g^2 + 6.3999\,g + 2.1067) + \xi g(5.0291 - 2.4g) + 0.49947$$
$$+ 0.122100\,\frac{g^2}{k^2} + \xi g(0.8g - 0.4743) + 0.057434\Big]$$

If σ is a compression, write

$$\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{I^2}{3E}\Big[\xi^2\left(20.7239\,g^4+6.3999\,g^3+2.1067\,g^2\right)+$$

$$-\ \xi\left(4.5562\,g-2.0\,g^2\right)+0.122100\,\frac{g^2}{k^2}+4.55730\Big]$$

Differentiate against $\frac{a}{\ell}$,

$$\xi^2\left(82.8956\,g^3+19.1999\,g^2+4.2134\,g\right)-\xi\left(4.5562-4.0\,g\right)$$

$$+\ 0.244200\,\frac{g}{k^2}=0$$

$$\cancel{\sim\quad k^2=\frac{0.244200\,g}{\xi\left(4.5562-4.0g\right)-\xi^2\left(82.8956\,g^3+19.1999\,g^2+4.2134\,g\right)}}$$

$$\cancel{\sim\qquad \left(4.5562-4g\right)=2\xi\left(82.8956\,g^3+19.1999\,g^2+4.2134\,g\right)}$$

$$\cancel{\therefore\quad \xi=\frac{1}{2}\,\frac{\left(4.5562-4g\right)}{\left(82.8956\,g^3+19.1999\,g^2+4.2134\,g\right)}}$$

$$\cancel{k^2=4\,\frac{0.244200\,g^2\left(82.8956\,g^2+19.1999\,g+4.2134\right)}{\left(4.5662-4g\right)^2}}$$

Now if we minus the energy expression with the quantity $\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{3E}$ 1, so that the expression truly represents the difference of total potential of the system in two modes, then

$$\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{3E}\Bigg[\left(\frac{E}{64\sigma}\right)^2\left(\frac{a}{R}\right)^4\left\{20.7239 f^4\left(\frac{a}{R}\right)^8 + 6.3999 f^3\left(\frac{a}{R}\right)^6 + 2.1067 f^2\left(\frac{a}{R}\right)^4\right\}$$

$$-\left(\frac{E}{64\sigma}\right)\left(\frac{a}{R}\right)^2\left\{4.5562 f\left(\frac{a}{R}\right)^2 - 2f^2\left(\frac{a}{R}\right)^4\right\} + 0.122100 f^2\left(\frac{a}{R}\right)^4\frac{1}{K^2} - 0.44270\Bigg]$$

the minimum condition becomes

$$\left(\frac{E}{64\sigma}\right)^2\left(\frac{a}{R}\right)^4\left\{82.8956 f^3\left(\frac{a}{R}\right)^8 + 19.1997 f^2\left(\frac{a}{R}\right)^6 + 4.2134 f\left(\frac{a}{R}\right)^4\right\}$$

$$-\left(\frac{E}{64\sigma}\right)\left(\frac{a}{R}\right)^2\left\{4.5562\left(\frac{a}{R}\right)^2 - 4f\left(\frac{a}{R}\right)^4\right\} + 0.244200 f\left(\frac{a}{R}\right)^4\frac{1}{K^2} = 0$$

or if $\left(\frac{a}{R}\right)\neq 0$

$$\boxed{\left(\frac{E}{64\sigma}\right)^2\left\{82.8956 f^3\left(\frac{a}{R}\right)^4 + 19.1999 f^2\left(\frac{a}{R}\right)^2 + 4.2134\right\}\left(\frac{a}{R}\right)^4 - \left(\frac{E}{64\sigma}\right)\left(\frac{a}{R}\right)^2\left\{4.5562 - 4f\left(\frac{a}{R}\right)^2\right\} + f\left(\frac{a}{R}\right)^2\frac{0.244200}{K^2} = 0}$$

$$\left(\frac{E}{640}\right)^2\left(\frac{a}{R}\right)^4\left\{145.0673 f^4\left(\frac{a}{R}\right)^8 + 38.3794 f^3\left(\frac{a}{R}\right)^6 + 10.5335 f^2\left(\frac{a}{R}\right)^4\right\}$$

$$-\left(\frac{E}{640}\right)\left(\frac{a}{R}\right)^2\left\{13.6686 f\left(\frac{a}{R}\right)^2 - 8.0 f^2\left(\frac{a}{R}\right)^4\right\} + 0.366300 f^2\left(\frac{a}{R}\right)^4\frac{1}{K^2} - 0.44210 = 0$$

Due to very nature of the conditions, it is easier to proceed as follows.

$$\left(\frac{a}{R}\right)^4\left\{82.8956 g^3 + 19.1999 g^2 + 4.2134 g\right\}\left(\frac{1}{64K\left(\frac{t}{R}\right)}\right)^2$$

$$-\left(\frac{a}{R}\right)^2\left\{4.5562 - 4g\right\}\frac{1}{64K\left(\frac{t}{R}\right)} + \frac{0.244200\, g}{K^2} = 0$$

$$\boxed{\left(\frac{a}{R}\right)^4 - \frac{64K\left(\frac{t}{R}\right)\left\{4.5562 - 4g\right\}}{\left\{82.8956 g^2 + 19.1999 g + 4.2134\right\} g}\left(\frac{a}{R}\right)^2 + \frac{0.244200 \times 64^2 \times \left(\frac{t}{R}\right)^2}{\left\{82.8956 g^2 + 19.1999 g + 4.2134\right\}} = 0}$$

$$\left(\frac{a}{R}\right)^4\left\{145.0673 g^4 + 38.3794 g^3 + 10.5335 g^2\right\}\left(\frac{1}{64K\left(\frac{t}{R}\right)}\right)^2$$

$$-\left(\frac{a}{R}\right)^2\left\{13.6686 g - 4g^2\right\}\frac{1}{64K\left(\frac{t}{R}\right)} + \frac{0.366300\, g^2}{K^2} - 0.44210 = 0$$

$$\boxed{\left(\frac{a}{R}\right)^4 - \frac{64K\left(\frac{t}{R}\right)\left\{13.6686 - 4g\right\}\left(\frac{a}{R}\right)^2}{\left\{145.0673 g^2 + 38.3794 g + 10.5335\right\} g} + \frac{0.366300 \times 64^2 \times \left(\frac{t}{R}\right)^2}{\left\{145.0673 g^2 + 38.3794 g + 10.5335\right\}} - \frac{0.44210 \times 64^2 \times K^2\left(\frac{t}{R}\right)^2}{\left\{145.0673 g^2 + 38.3794 g + 10.5335\right\} g^2} = 0}$$

Both equations can be written as

$$\left(\frac{a}{R}\right)^4 - \frac{K\left(\frac{t}{R}\right)\left\{1.13905-g\right\}}{\underbrace{\left\{0.32381\,g^2+0.075000\,g+0.01459\right\}}_{X}g}\left(\frac{a}{R}\right)^2 + \frac{3.9072\left(\frac{t}{R}\right)^2}{\left\{0.32381\,g^2+0.075000\,g+0.01459\right\}} = 0$$

$$\left(\frac{a}{R}\right)^4 - \frac{K\left(\frac{t}{R}\right)\left\{1.70858-g\right\}}{\underbrace{\left\{0.28333\,g^2+0.075000\,g+0.020523\right\}}_{W}g}\left(\frac{a}{R}\right)^2 + \frac{\left(2.9304\,g^2-3.5416\,K^2\right)\left(\frac{t}{R}\right)^2}{\left\{0.28333\,g^2+0.075000\,g+0.020523\right\}g^2} = 0$$

$$\left(\frac{a}{R}\right)^2 = \frac{1}{2}\,\frac{K\left(\frac{t}{R}\right)\left\{1.13905-g\right\}}{X g} \pm \sqrt{\frac{1}{4}\,\frac{K^2\left(\frac{t}{R}\right)^2\left\{1.13905-g\right\}^2}{X^2 g^2} - \frac{3.9072\left(\frac{t}{R}\right)^2}{X}}$$

$$= \frac{1}{2}\,\frac{K\left(\frac{t}{R}\right)\left\{1.13905-g\right\}}{X g}\left[1 \pm \sqrt{1 - \frac{4\times 3.9072\,X g^2}{K^2\left\{1.13905-g\right\}^2}}\right]$$

$$\frac{K\left(\frac{t}{R}\right)^2}{g^2}\left[\frac{(1.70858-g)}{W} - \frac{(1.13905-g)}{X}\right]\frac{1}{2}\,\frac{(1.13905-g)}{X}\left[1 \pm \sqrt{1 - \frac{4\times 3.9072\,X g^2}{K^2(1.13905-g)^2}}\right]$$

$$+ \frac{\left(\frac{t}{R}\right)^2}{g^2}\left[\frac{3.9072\,g^2}{X} - \frac{\left(2.9304\,g^2-3.5416\,K^2\right)}{W}\right] = 0$$

$$\frac{1}{2}\frac{(1.13905-g)}{X}\left[\frac{(1.70858-g)}{W}-\frac{(1.13905-g)}{X}\right]\left[1\pm\sqrt{1-\frac{15.6288\,X g^2}{K^2(1.13905-g)^2}}\right]K^2$$

$$+\left[\frac{3.9072g^2}{X}-\frac{(2.9304g^2-3.5416K^2)}{W}\right]=0$$

where $X = 0.32381g^2 + 0.075000g + 0.016459$

$W = 0.48333g^2 + 0.075000g + 0.020573$

When $g = 0.1$

$X = 0.027197$, $W = 0.030906$ $\quad \frac{1}{X} = 36.769$

$\frac{1}{W} = 32.356$

$$\frac{1}{2}\frac{1.03905}{0.027197}\left[\frac{1.60858}{0.030906}-\frac{1.03905}{0.027197}\right]\left[1\pm\sqrt{1-\frac{0.156288\times 0.027197}{K^2\times 1.03905^2}}\right]K^2$$

$$=\frac{0.029304-3.5416K^2}{0.030906}-\frac{0.039072}{0.027197}$$

$$1-\frac{0.156288\times 0.027197}{K^2\times 1.03905^2}=\left[\frac{32.356}{19.1024\times 13.8424}\left(\frac{0.029304}{K^2}-3.5416\right)-\frac{36.769\times 0.039072}{19.1024\times 13.8424}\frac{1}{K^2}-1\right]^2$$

$$1-\frac{0.0039371}{K^2}=\left[0.12237\left(\frac{0.029304}{K^2}-3.5416\right)-\frac{0.0054331}{K^2}-1\right]^2$$

$$=\left[\frac{0.0018412}{K^2}+0.56161\right]^2$$

$$1 - \frac{0.0039371}{K^2} = \left(\frac{0.0018429}{K^2}\right)^2 + \frac{0.0020933}{K^2} + 0.32105$$

$$\left(\frac{0.001}{K^2}\right)^2 3.41215 + \left(\frac{0.001}{K^2}\right) 6.0304 - 0.67895 = 0$$

$$\left(\frac{0.001}{K^2}\right)^2 + 1.7673\left(\frac{0.001}{K^2}\right) - 0.19898 = 0$$

$$\left(\frac{0.001}{K^2}\right) = -0.88365 + \sqrt{0.88365^2 + 0.19898}$$

$$= -0.88365 + \sqrt{0.97982} = -0.88365 + 0.98986$$

$$= 0.10621$$

$$\therefore \quad K^2 = 0.0094153 \qquad \boxed{k = 0.097032}$$

$$\frac{\left(\frac{a}{R}\right)^2}{\left(\frac{t}{R}\right)} = \frac{1}{2}\,\frac{0.97032 \times 1.03905}{0.027197}\left[1 \pm \sqrt{1 - \frac{4 \times 3.9072 \times 0.027197 \times 1.0621}{1.03905^2}}\right]$$

$$= 18.3845 \times 0.97032 \times 1.03905\left[1 \mp \sqrt{0.58173}\right]$$

0.23722
1.76273

$$= 4.397$$

32.674

If $\left(\frac{t}{R}\right) = \frac{1}{1000}$ $\qquad \left(\frac{a}{R}\right)^2 = 0.004397 \qquad \boxed{\frac{a}{R} = 0.01135}$

$\cancel{f\left(\frac{a}{R}\right)^2 = 0.1}$ $\qquad \boxed{\cancel{f = \frac{0.1}{0.032674} = 3.060}}$

3.2.674 $\qquad$ 0.8169

$$\frac{w_{max}}{t} = f\,\frac{\left(\frac{a}{R}\right)^4}{4}\,\frac{\rho}{t} = \frac{g\left(\frac{a}{R}\right)^2/(t/R)}{4} = \frac{0.1 \times 4.397}{4} = \underline{\underline{0.1099}}$$

If we consider (ξg) also as a variable,

$$6q_2 + 4s_2 - \tfrac{1}{2} = 2\xi g\, p_2 + \tfrac{1}{3}\xi g$$

$$6q_2 - \tfrac{1}{3} = \xi g\, p_2 + 12\xi g\, r_2 - \xi g$$

$$-6q_2 - 2s_2 - \tfrac{1}{2} = 2\xi g\, p_2 + 6\xi g\, r_2 - \tfrac{5}{3}\xi g$$

$$-0.65 - 2.6q_2 - 4s_2 = 2.3\,\xi g\, p_2 + 1.2\,\xi g\, r_2 + 4.5\,\xi g$$

$$2.6q_2 - 0.65 = 1.95\,\xi g\, p_2 + 6.6\,\xi g\, r_2 - 0.2\,\xi g$$

This is a system of equations for 5 unknowns, $q_2, s_2, p_2, r_2, (\xi g)$

$$\begin{cases} q_2 + 0.666667\, s_2 - 0.33333\,\xi g\, p_2 + 0 = 0.055556\,\xi g + 0.083333 \\ q_2 + 0 - 0.166667\,\xi g\, p_2 - 2\xi g\, r_2 = -0.833333\,\xi g + 0.083333 \\ q_2 + 0.333333\, s_2 + 0.33333\,\xi g\, p_2 + \xi g\, r_2 = +0.277778\,\xi g - 0.083333 \\ q_2 + 1.538465\, s_2 + 0.88461\,\xi g\, p_2 + 0.46154\,\xi g\, r_2 = -1.73077\,\xi g - 0.25000 \\ q_2 + 0 - 0.75000\,\xi g\, p_2 - 2.53846\,\xi g\, r_2 = -0.076923\,\xi g + 0.25000 \end{cases}$$

$$0.666667\, s_2 - 0.166667\,\xi g\, p_2 + 2\xi g\, r_2 = 0.888889\,\xi g$$

$$0.33333\, s_2 + 0.5000\,\xi g\, p_2 + 3\xi g\, r_2 = 1.111111\,\xi g - 0.166667$$

$$1.20513\, s_2 + 0.55128\,\xi g\, p_2 - 0.53846\,\xi g\, r_2 = -2.00855\,\xi g - 0.16667$$

$$1.53846\, s_2 + 1.63461\,\xi g\, p_2 + 3.0000\,\xi g\, r_2 = -1.65385\,\xi g - 0.50000$$

$$\delta_2 - 0.25000\,\xi_1^q f_2 + 3.0000\,\xi_1^q h_2 = 1.333\ldots\,\xi_1^q$$
$$\delta_2 + 1.5000\,\xi_1^q f_2 + 9.000\,\xi_1^q h_2 = \ldots\,\xi_1^q - 0.5000$$
$$\delta_2 + 0.45244\,\xi_1^q f_2 - 0.44681\,\xi_1^q h_2 = -1.11167\,\xi_1^q - 0.138298$$
$$\delta_2 + 1.06250\,\xi_1^q f_2 + 1.950\,\xi_1^q h_2 = -1.07500\,\xi_1^q - 0.32500$$

$$1.75000\,\xi_1^q f_2 + 6.0000\,\xi_1^q h_2 = 2.00000\,\xi_1^q - 0.50000$$
$$1.04256\,\xi_1^q f_2 + 9.44681\,\xi_1^q h_2 = 5.0000\,\xi_1^q - 0.36170$$
$$0.60506\,\xi_1^q f_2 + 2.39681\,\xi_1^q h_2 = 0.59167\,\xi_1^q - 0.18170$$

$$\xi_1^q f_2 + 3.42857\,\xi_1^q h_2 = 1.14286\,\xi_1^q - 0.28571$$
$$\xi_1^q f_2 + 9.06116\,\xi_1^q h_2 = 4.29779\,\xi_1^q - 0.34693$$
$$\xi_1^q f_2 + 3.96128\,\xi_1^q h_2 = 0.97282\,\xi_1^q - 0.30656$$

$$5.63259\,\xi_1^q h_2 = 3.65303\,\xi_1^q - 0.06122$$
$$5.09988\,\xi_1^q h_2 = 3.81809\,\xi_1^q - 0.03837$$

$$4.21185\,\xi_1^q - 0.04238 = 3.65303\,\xi_1^q - 0.06122$$

$$\xi_1^q = -\frac{0.01884}{0.56382} = -0.03341$$

If ... taken as ...

$$\boxed{\xi_1^q = 0.03344}$$

$$\boxed{\xi_1^q = 0.03341}$$

$$r_2 = \frac{-7.47105 \times 0.03341 - 0.09959}{-10.73247 \times 0.03341}$$

$$= \frac{0.09959 + 0.24961}{0.35857} = \boxed{0.97367 = r_2}$$

$$p_2 = \frac{-0.03341(-16.02115 + 6.91662) - 0.94120}{-3 \times 0.03341}$$

$$= -3.03484 + 9.39039 = \boxed{6.35555 = p_2}$$

$$s_2 = \frac{-0.03341(-17.60447 - 13.15035 + 17.25000) - 0.96330}{4}$$

$$= 0$$

$$q_2 = \frac{-0.03341(0.20376 + 2.99652 - 2.30468) + 0.083333}{5}$$

$$= \boxed{0.01070 = q_2}$$

Check $\quad 0.01070 + 0.33333 \times 0.03341 \times 6.3555$

$$= 0.08148$$

$$q_0 = \frac{16}{3}g - 8 + \frac{1}{[illegible]} = \frac{16}{3}g - 37.931$$

$$r_0 = \xi g\left(\frac{2}{3}g - \frac{4}{3}\right)$$

$$q_2 = 0.01070$$

$$s_2 = 0$$

$$p_2 = 6.2555$$

$$r_2 = 0.97341$$

$$\xi g = -0.03341$$

$$\xi g = -0.03341$$

$$= \frac{E}{640}\left(\frac{a}{R}\right)^2 g$$

$$\left(\frac{a}{R}\right)^2 = -\frac{640 \times 0.03341}{g}\left(\frac{\sigma}{E}\right)$$

$$\frac{\mathcal{E}_1}{r^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[2.6\left\{\frac{4}{9}(\xi g)^2(g^2 - 4g + 4) + 12 \times 0.01070^2\right\}\right]$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[2.6\left\{0.0004961\,(g^2 - 4g + 4) + 0.0013371\right\}\right]$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[2.6\left\{0.0004961\,g^2 - 0.0019844\,g + 0.0033[illegible]\right\}\right]$$

$$\boxed{\frac{\mathcal{E}_1}{r^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[0.0012897\,g^2 - 0.0051594\,g + 0.0086310\right]}$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 (0.03341)^2 \Big[0.35\,(5.3333\,g - 37.931)^2 + (37.333 - 28.000\,g)$$
$$\times (5.3333\,g - 37.931)$$

$$+ 15.6444 - 31.2889\,g + 37.5316\,g^2 + 6.9125$$

$$+ 13 \times 6.3555^2 + 2.2 \times 6.\ldots \times 0.97367 - 53 \times 6.3555 + 22.05 \times 0.97367$$
$$+ 7.7 \times 0.97367^2 \Big]$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 (0.03341)^2 \Big[$$

g^2	g	
$+9.95555$	-141.609	$+503.536$
-14.73333	$+106.207$	-141.609
$+37.5316$	$+19.911$	$+15.644$
	-31.289	$+6.9\ldots$
		$+335.\ldots$
		$+155.\ldots$
		$-336.8\ldots$
		$+7.018$
		$+21.474$
32.7538	-46.780	872.553

$$\boxed{\begin{aligned}
\frac{\mathcal{E}_2}{R^3} &= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[0.032568\,g^2 - 0.052216\,g + 0.97394\Big]\\
\frac{\mathcal{E}_3}{R^3} &= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\;0.1220\;\frac{g}{\left(\frac{\sigma}{E}\frac{R}{t}\right)^2}\Big]\\
\frac{\delta\mathcal{E}}{R^3} &= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\Big[\ldots + 0.013367\,g - 0.021733\Big]\times a^2
\end{aligned}}$$

?

$$0.033798\,g^2 - 0.084109\,g + 0.03604 + \frac{0.12210\,g^2}{K^2} = 0$$

$$g^2 - 2.4886\,g + 1.0663 = 0$$

$$g = 1.2443 \pm \sqrt{1.2443^2 - 1.0663} = 1.2443 \pm \sqrt{0.4820} = 1.2443 \pm 0.143$$

$$= 2.5500$$

1.9386

$$\boxed{K^2 = \frac{g^2}{0.68865\,g - (0.29517 + 0.22461\,g^2)}}$$

①	②	③	④	⑤	⑥	⑦	⑧
g	g^2	$0.68865\,g$	$0.22461\,g^2$	③ − (0.29 ... ④)	②/⑤ = K^2		
0.60	0.36	0.41331	0.09765	0.01849	9.47		
0.70	0.49	0.48220	0.13564	0.05139	9.53		
0.80	0.64	0.55106	0.2216	0.07825	8.13		
0.90	0.81	0.61997	0.22402	0.10058	8.05		
1.00	1.00	0.68865	0.22561	0.11647	8.5		
1.10	1.21	0.75774	0.33494	0.12763	9.49		
1.20	1.44	0.82662	0.39861	0.13284	10.84		
1.40	1.96	0.96439	0.46781	0.20141			
1.60	2.56	1.10216	0.20763	0.09836			
1.80	3.24	1.23993	[illegible]	0.04390			

Comparison of different energies

For $g = 0.90$,

(Difference!)

$\frac{E_1}{t^3}$ = extensional strain energy outside the circular reg[illegible]

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[\begin{array}{r} 0.001044 \\ -0.004640 \\ 0.008638 \\ \hline 0.005042 \end{array}\right]$$

$$\xi g = -0.03341$$

$$\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2 g = -0.03341$$

$$\left(\frac{a}{R}\right)^2 = \frac{-64 \times 0.03341}{g}\frac{\sigma}{E}$$

For the circular region

$$\widehat{rr} = \frac{Ef}{64}\Big[\Big\{\frac{1}{2}Q_0 + 4\left(\frac{a}{\rho}\right)^2\left(1 - f\frac{a^2}{c^2}\right)\left(\frac{\rho}{c}\right)^2 + \frac{4}{3}\left(3f\frac{a^2}{c^2} - 1\right)\left(\frac{\rho}{c}\right)^4 - \frac{2}{3}f\left(\frac{\rho}{c}\right)^6\Big\}$$
$$- \cos 2\theta\Big\{2P_2 + \frac{1}{3}\left(\frac{\rho}{c}\right)^4\Big\}\Big]$$

$$\widehat{\theta\theta} = \frac{Ef}{64}\Big[\Big\{\frac{1}{2}Q_0 + 12\frac{a^2}{c^2}\left(1 - f\frac{a^2}{c^2}\right)\left(\frac{\rho}{c}\right)^2 + \frac{20}{3}\left(3f\frac{a^2}{c^2} - 1\right)\left(\frac{\rho}{c}\right)^4 - \frac{14}{3}f\left(\frac{\rho}{c}\right)^6\Big\}$$
$$+ \cos 2\theta\Big\{2P_2 + [illegible]\,P_2\left(\frac{\rho}{c}\right)^2 - 5\left(\frac{\rho}{c}\right)^4\Big\}\Big]$$

$$\widehat{r\theta} = \frac{Ef}{64}\Big[\sin 2\theta\Big\{2P_2 + 6R_2\left(\frac{\rho}{c}\right)^2 - \frac{5}{3}\left(\frac{\rho}{c}\right)^4\Big\}\Big]$$

$$\frac{1}{E}\left(\widehat{rr} - \nu\widehat{\theta\theta}\right) = \frac{f}{64}\Big[\Big\{(1-\nu)\frac{Q_0}{2} + 4(1-3\nu)\frac{a^2}{c^2}\left(1 - f\frac{a^2}{c^2}\right)\left(\frac{\rho}{c}\right)^2 + \frac{4}{3}(1-5\nu)\left(3f\frac{a^2}{c^2} - 1\right)\left(\frac{\rho}{c}\right)^4$$
$$- \frac{2}{3}(1-7\nu)f\left(\frac{\rho}{c}\right)^6\Big\} - \cos 2\theta\Big\{2(1+\nu)P_2 + 12\nu P_2\left(\frac{\rho}{c}\right)^2 + \left(\frac{1}{3} - 5\nu\right)\left(\frac{\rho}{c}\right)^4\Big\}\Big]$$

$$\frac{1}{2}\Big\{\left(\frac{\partial w}{\partial r}\right)^2 - \left(\frac{\partial w_0}{\partial r}\right)^2\Big\} = \frac{f}{64}\Big\{32\left(\frac{a}{\rho}\right)^2\left(f\frac{a^2}{c^2} - 1\right)\frac{\rho^2}{c^2} - 32\left(3f\frac{a^2}{c^2} - 1\right)\left(\frac{\rho}{c}\right)^4 + 32f\left(\frac{\rho}{c}\right)^6$$
$$+ \cos 2\theta\left(32\left(\frac{a}{c}\right)^2 - 32\left(\frac{\rho}{c}\right)^2\right)\frac{\rho^2}{c^2}\Big\}$$

Therefore

$$\frac{\partial u}{\partial r} = \frac{f}{64}\Big[\Big\{(1-\nu)\frac{Q_0}{2} + 12(3-\nu)\frac{a^2}{c^2}\left(1 - f\frac{a^2}{c^2}\right)\left(\frac{\rho}{c}\right)^2 + \frac{20}{3}(5-\nu)\left(3f\frac{a^2}{c^2} - 1\right)\left(\frac{\rho}{c}\right)^4$$
$$- \frac{14}{3}(7-\nu)f\left(\frac{\rho}{c}\right)^6\Big\} - \cos 2\theta\Big\{2(1+\nu)P_2 + 4\left(8\frac{a^2}{c^2} + 3\nu P_2\right)\left(\frac{\rho}{c}\right)^2$$
$$- 5\left(\frac{19}{3} + \nu\right)\left(\frac{\rho}{c}\right)^4\Big\}\Big]$$

$$\frac{u}{R} = \frac{p}{64}\Big[\Big\{\frac{(1-\nu)}{2}Q_0\left(\frac{a}{R}\right) + 4(3-\nu)\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{a}{R}\right)^3 + \frac{4}{3}(5-\nu)\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{a}{R}\right)^5$$
$$- \frac{2}{3}(7-\nu)f\left(\frac{a}{R}\right)^7\Big\} - \cos 2\theta\Big\{2(1+\nu)P_2\left(\frac{a}{R}\right) + \frac{4}{3}\left(f\frac{a^2}{R^2} + 3\nu P_2\right)\left(\frac{a}{R}\right)^3$$
$$- \left(\frac{13}{3}+\nu\right)\left(\frac{a}{R}\right)^5\Big\}\Big]$$

$$\frac{1}{E}(\hat{\sigma}_r - \nu\hat{\sigma}_\theta) = \frac{p}{4}\Big[\Big\{\frac{(1-\nu)}{2}Q_0 + 4(3-\nu)\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{a}{R}\right)^2 + \frac{4}{3}(5-\nu)\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{a}{R}\right)^4$$
$$- \frac{2}{3}(7-\nu)f\left(\frac{a}{R}\right)^6\Big\} + \cos 2\theta\Big\{2(1+\nu)P_2 + [illegible]\, P_2\left(\frac{a}{R}\right)^2 - \left(5 - \frac{1}{3}\nu\right)\left(\frac{a}{R}\right)^4\Big\}\Big]$$

$$\frac{1}{E}\hat{\tau} = \frac{p}{4}\cos 2\theta\Big\{4(1+\nu)P_2 + \frac{32}{3}f\left(\frac{a}{R}\right)^2 + 12(1+\nu)P_2\left(\frac{a}{R}\right)^2 - \frac{8}{3}(17+\nu)\left(\frac{a}{R}\right)^4\Big\}$$

$$\boxed{\frac{v}{R} = \frac{p}{64}\sin 2\theta\Big\{2(1+\nu)P_2\left(\frac{a}{R}\right) + \frac{16}{3}f\left(\frac{a}{R}\right)^3 + 4(1+\nu)P_2\left(\frac{a}{R}\right)^3 - \frac{4}{3}(17+\nu)\left(\frac{a}{R}\right)^5\Big\}}$$

2. [illegible] outside the circle

$$\hat{\sigma}_r = p\Big[\frac{1}{2} + \frac{[illegible]}{\left(\frac{a}{R}\right)^2} + \cos 2\theta\Big\{\frac{1}{2} - \frac{[illegible]}{\left(\frac{a}{R}\right)^4} - \frac{4Q_2}{\left(\frac{a}{R}\right)^2}\Big\}\Big]$$

$$\hat{\sigma}_\theta = p\Big[\frac{1}{2} - \frac{P_0}{\left(\frac{a}{R}\right)^2} + \cos 2\theta\Big\{\frac{6Q_2}{\left(\frac{a}{R}\right)^4} - \frac{1}{2}\Big\}\Big]$$

$$\hat{\tau} = -p\sin 2\theta\Big\{\frac{1}{2} + \frac{[illegible]}{\left(\frac{a}{R}\right)^4} + \frac{2Q_2}{\left(\frac{a}{R}\right)^2}\Big\}$$

$$\frac{u}{R} = \frac{\gamma}{E}\left[\frac{1}{2}(1-\nu)\left(\frac{r}{R}\right) - (1+\nu)\frac{P_0}{\left(\frac{r}{R}\right)} + \cos 2\theta\left\{\frac{1}{2}(1+\nu)\left(\frac{r}{R}\right) + 2(1+\nu)\frac{Q_2}{\left(\frac{r}{R}\right)^3} + \frac{4S_2}{\left(\frac{r}{R}\right)}\right\}\right]$$

$$\frac{v}{R} = \frac{\gamma}{E}\sin 2\theta\left\{2(1+\nu)\frac{Q_2}{\left(\frac{r}{R}\right)^3} - \frac{1}{2}(1+\nu)\frac{r}{R}\right\}$$

With the simplified relations of p 316, the conditions of strain continuity becomes

$$\frac{1}{2} + n_0 = 5g\left\{\frac{1}{2}q_0 + 4(1-g) + \frac{2}{3}(2g-1) - \frac{2}{3}g\right\}$$

$$-\frac{1}{2} + 6q_2 + 4S_2 = 5g\left\{2p_2 + \frac{1}{3}\right\}$$

$$\frac{1}{2} - n_0 = 5g\left\{\frac{1}{2}q_0 + 12(1-g) + \frac{20}{3}(2g-1) - \frac{14}{3}g\right\}$$

$$6q_2 - \frac{1}{2} = 5g\left\{2p_2 + 12n_2 - 5\right\}$$

$$-\frac{1}{2} - 6q_2 - 2S_2 = 5g\left\{2p_2 + 6n_2 - \frac{5}{3}\right\}$$

$$\frac{1}{2}(1+\nu) + 2(1+\nu)q_2 + 4S_2 = 5g\left\{-2(1+\nu)p_2 - \frac{4}{3}(1+3\nu n_2) + \left(\frac{13}{3}+\nu\right)\right\}$$

$$2(1+\nu)q_2 - \frac{1}{2}(1+\nu) = 5g\left\{2(1+\nu)p_2 + 6(1+\nu)n_2 - \frac{5}{3}(1+\nu)\right\}$$

Thus
$$\boxed{\begin{aligned} p_0 &= \frac{1}{g5} - 6\left(1 - \frac{2}{3}g\right) \\ n_0 &= 5g\,\frac{2}{3}\,(g-2) \end{aligned}}$$

$$q_2 + 0.666667\, s_2 - 0.083333 = \xi g \{0.33333\, p_2 + 0.055556\}$$

$$q_2 + 0 \quad - 0.083333 = \xi g \{0.33333\, p_2 + 2 r_2 - 0.833333\}$$

$$-q_2 - 0.333333\, s_2 - 0.083333 = \xi g \{0.33333\, p_2 + r_2 - 2.277778\}$$

$$q_2 + 1.53846\, s_2 + 0.25000 = \xi g \{- p_2 - 0.461538\, r_2 - 1.55128\}$$

$$q_2 \quad - 0.25000 = \xi g \{p_2 + 3 r_2 - 0.155128\}$$

$$0.666667\, s_2 + 0 = \xi g \{-2 r_2 + 0.888889\}$$

$$-0.33333\, s_2 - 0.1666667 = \xi g \{0.66667\, p_2 + 3 r_2 - 1.111111\}$$

$$1.20513\, s_2 + 0.166667 = \xi g \{-0.66667\, p_2 + 0.538462\, r_2 - 1.82906\}$$

$$1.53846\, s_2 + 0.50000 = \xi g \{-2 p_2 - 3.461538\, r_2 - 1.39615\}$$

$$s_2 + 0 = \xi g \{-3 r_2 + 1.33333\}$$

$$-s_2 - 0.5000 = \xi g \{2 p_2 + 9 r_2 - 3.33333\}$$

$$s_2 + 0.138298 = \xi g \{-0.553191\, p_2 + 0.446808\, r_2 - 1.51773\}$$

$$s_2 + 0.325 = \xi g \{-1.3\, p_2 - 2.25000\, r_2 - 0.907498\}$$

$$2 \xi g\, p_2 + 6 \xi g\, r_2 = 2 \xi g - 0.5000$$

$$1.446809\, \xi g\, p_2 + 9.446808\, \xi g\, r_2 = 4.85106\, \xi g - 0.361702$$

$$0.746809\, \xi g\, p_2 + 2.196808\, \xi g\, r_2 = 0.61023\, \xi g - 0.186702$$

$$\Sigma_g p_2 + 3\Sigma_g r_2 = \Sigma_g - 0.250000$$

$$\Sigma_g p_2 + 6.52941\,\Sigma_g r_2 = 3.35294\,\Sigma_g - 0.25000$$

$$\Sigma_g p_2 + 3.61111\,\Sigma_g r_2 = 0.817116\,\Sigma_g - 0.25000$$

$$3.52941\,\Sigma_g r_2 = 2\,\text{[illegible]} + \Sigma_g$$

$$2.91833\,\Sigma_g r_2 = 2.535\text{[illegible]}\,\Sigma_g$$

Bud [illegible]

Part 2 of Least Square

$$\boxed{q_{12} + 0.337492\,s_2 - 0.0166667 = \Sigma_g\{0.06\text{[illegible]}\,p_2 + 0.707692\,r_2 - 0.441261\}}$$

$$1.500\,q_{12} + s_2 - 0.12500 = \Sigma_g\{0.\text{[illegible]}\,p_2 + 0.\text{[illegible]}\}$$

$$3\,q_{12} + s_2 + \text{[illegible]} = \Sigma_g\{-\,\text{[illegible]} - 3r_2 + \text{[illegible]}\}$$

$$0.65\,q_2 + s_2 + 0.1625 = \Sigma_g\{-0.65\,p_2 - 0.3000\,r_2 - 1.00833\}$$

$$\boxed{1\,q_{12} + 0.582525\,s_2 + 0.0758253 = \Sigma_g\{-0.223301\,p_2 - 0.640778\,r_2 - 0.0173494\}}$$

$$0.166667\, q_2 + 0.444444\, s_2 - 0.055556 = \frac{\xi q}{\ell}\left\{0.222222\, f_2 + 0.0370371\right\}$$

$$0.333333\, q_2 + 0.1111111\, s_2 + 0.0277777 = \frac{\xi q}{\ell}\left\{-0.1111\, f_2 - 0.33333\, a_2 + 0.0925926\right\}$$

$$1.53846\, q_2 + 2.36686\, s_2 + 0.3[illegible] = \frac{\xi q}{\ell}\left\{-1.5[illegible]\, f_2 - 0.7100[illegible]\, a_2 - 2.38658\right\}$$

$$2.53[illegible]6\, q_2 + 2.[illegible]2241\, s_2 + 0.[illegible] = \frac{\xi q}{\ell}\left\{-1.42[illegible]\, f_2 - [illegible]\, a_2 - 2.25[illegible]\right\}$$

$$\boxed{q_2 + 1.1[illegible]\, s_2 + 0.14[illegible]73 = \frac{\xi q}{\ell}\left\{-[illegible]\, f_2 - 0.411032\, a_2 - 0.589101\right\}}$$

$$0.[illegible]\, q_2 - 0.222[illegible]\, s_2 - 0.0277777 = \frac{\xi q}{\ell}\left\{0.11111\, f_2 + 0.018518[illegible]\right\}$$

$$0.33333\, q_2 \qquad - 0.0277777 = \frac{\xi q}{\ell}\left\{0.11111\, f_2 + 0.666667\, a_2 - 0.[illegible]\right\}$$

$$-[illegible]\, q_2 - 0.11[illegible]\, s_2 [illegible] = \frac{\xi q}{\ell}\left\{[illegible]\, f_2 - [illegible]\, a_2 - [illegible]\right\}$$

$$q_2 - 1.53846\, s_2 - [illegible] = \frac{\xi q}{\ell}\left\{f_2 + 0.461538\, a_2 + [illegible]\right\}$$

$$q_2 \qquad - 0.[illegible] = \frac{\xi q}{\ell}\left\{f_2 + 3\, a_2 - 0.1[illegible]\right\}$$

$$0.333333\, q_2 - 1.42735\, s_2 - 0.583333 = \frac{\xi q}{\ell}\left\{2.33333\, f_2 + 4.461534\, a_2 + 1.04430\right\}$$

$$\boxed{q_2 - 4.28205\, s_2 - 1.75000 = \frac{\xi q}{\ell}\left\{7\, f_2 + 13.3846\, a_2 + 3.13290\right\}}$$

$$2 q_{f2} + 0 \quad - 0.1666667 = \xi \frac{g}{l} \left\{ 0.666667 f_2 + 4 r_2 - 1.666667 \right\}$$

$$- q_{f2} - 0.33333 s_2 - 0.013333 = \xi \frac{g}{l} \left\{ 0.33333 f_2 + r_2 - 0.277778 \right\}$$

$$-0.461538 q_{f2} - 0.710056 s_2 - 0.115345 = \xi \frac{g}{l} \left\{ 0.461538 f_2 + 0.213[illegible] r_2 + 0.715[illegible] \right\}$$

$$3 q_{f2} \qquad - 0.25[illegible] = \xi \frac{g}{l} \left\{ 3 f_2 + 9 r_2 - 0.465384 \right\}$$

$$5.84[illegible] q_{f2} - 1.043391 s_2 - [illegible]345 = \xi \frac{g}{l} \left\{ 4.461538 f_2 + 14.213017 r_2 - 1.693854 \right\}$$

$$\boxed{q_{f2} - 0.294871 s_2 - 0.315218 = \xi \frac{g}{l} \left\{ 1.26067 f_2 + 4.01672 r_2 - 0.478698 \right\}}$$

The equations for constants are then

$$q_2 + 0.507692\, s_2 - 0.016667\, \xi g\, p_2 - 0.737192\, \xi g\, r_2 = -0.441581\, \xi g + 0.011667$$

$$q_2 + 1.15125\, s_2 + 0.562289\, \xi g\, p_2 + 2.311039\, \xi g\, r_2 = -0.889114\, \xi g - 0.640573$$

$$q_2 - 4.28205\, s_2 - 7.00000\, \xi g\, p_2 - 13.3166\, \xi g\, r_2 = 3.13290\, \xi g + 1.25000$$

$$q_2 - 0.294271\, s_2 - 1.26017\, \xi g\, p_2 - 4.01572\, \xi g\, r_2 = -0.478698\, \xi g + 0.315218$$

$$0.64356\, s_2 + 0.628956\, \xi g\, p_2 + 1.17724\, \xi g\, r_2 = -0.447820\, \xi g - 0.157240$$

$$5.43330\, s_2 + 7.562289\, \xi g\, p_2 + 13.79563\, \xi g\, r_2 = -4.02200\, \xi g - 1.890573$$

$$3.98778\, s_2 + 5.73913\, \xi g\, p_2 + 9.36788\, \xi g\, r_2 = -3.61160\, \xi g - 1.43478$$

$$s_2 + 0.977310\, \xi g\, p_2 + 1.73834\, \xi g\, r_2 = -0.695850\, \xi g - 0.244321$$

$$s_2 + 1.391839\, \xi g\, p_2 + 2.53909\, \xi g\, r_2 = -0.740249\, \xi g - 0.347960$$

$$s_2 + 1.439397\, \xi g\, p_2 + 2.34950\, \xi g\, r_2 = -0.905664\, \xi g - 0.357849$$

$$0.462087\, \xi g\, p_2 + 0.61116\, \xi g\, r_2 = -0.209954\, \xi g - 0.115520$$

$$0.414529\, \xi g\, p_2 + 0.80075\, \xi g\, r_2 = -0.044399\, \xi g - 0.103631$$

$$\xi g\, p_2 + 1.32261\, \xi g\, r_2 = -0.454359\, \xi g - 0.249996$$

$$\xi g\, p_2 + 1.93171\, \xi g\, r_2 = -0.107107\, \xi g - 0.249996$$

$$0.60910\, \xi g\, r_2 = 0.347252\, \xi g$$

$$\boxed{\xi_\varrho \lambda_2 = 0.570106\,\xi_\varrho}$$

$$2\,\xi_\varrho \mu_2 = -2.61578\,\xi_\varrho - 2 \times 0.25000$$

$$\boxed{\xi_\varrho \mu_2 = -1.30789\,\xi_\varrho - 0.25000}$$

$$3\,\delta_2 = (4.60233 - 3.77805 - 2.34190\ )\,\xi_\varrho + 0.952163 - 0.952138$$

$$\boxed{\delta_2 = -0.505873\,\xi_\varrho}$$

$$\gamma_2 = \xi_\varrho(-1.77613 - 9.31345 + 10.08992 + 1.32372) - 194131 + 194131$$

$$\boxed{\gamma_2 = 0.138490\,\xi_\varrho}$$

Chck. ξ_ϱ +0.138490
−0.256828
+0.080559 + 0.0166667
−0.403459

−0.441238

O.K

The dimensional strain energy in the circular region

$$\hat{V}_i + \hat{V}_3 = \frac{Ef}{64}\left[\left\{ P_0 + 16\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{a}{R}\right)^2 + 8\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{a}{R}\right)^4 - \frac{16}{3}f\left(\frac{a}{R}\right)^6 \right\}\right.$$

$$\left. + \cos 2\theta \left\{ 12 P_2 \left(\frac{a}{R}\right)^2 - \frac{16}{3}\left(\frac{a}{R}\right)^4 \right\}\right]$$

$$- \left\{ 2P_2 + \frac{4}{3}\left(\frac{a}{R}\right)^4 \right\}\left\{ 2P_2 + 12P_2\left(\frac{a}{R}\right)^2 - 5\left(\frac{a}{R}\right)^4 \right\} - \left\{ 2P_2 + 6P_2\left(\frac{a}{R}\right)^2 - \frac{5}{3}\left(\frac{a}{R}\right)^4 \right\}^2$$

$$- \left[4P_2^2 + \frac{2}{3}P_2\left(\frac{a}{R}\right)^4 + 24 P_2 P_2 \left(\frac{a}{R}\right)^2 + 4P_2\left(\frac{a}{R}\right)^4 - 10 P_2\left(\frac{a}{R}\right)^4 - \frac{5}{3}\left(\frac{a}{R}\right)^6 \right]$$

$$- \left[4P_2^2 + 36 P_2^2\left(\frac{a}{R}\right)^4 + \frac{25}{9}\left(\frac{a}{R}\right)^8 + 24 P_2 P_2 \left(\frac{a}{R}\right)^2 - \frac{20}{3}P_2\left(\frac{a}{R}\right)^4 - 20 P_2\left(\frac{a}{R}\right)^6 \right]$$

$$= - \left[8P_2^2 + 36 P_2^2\left(\frac{a}{R}\right)^4 + 48 P_2 P_2 \left(\frac{a}{R}\right)^2 - 16 P_2\left(\frac{a}{R}\right)^2 - 16 P_2\left(\frac{a}{R}\right)^6 + \frac{10}{9}\left(\frac{a}{R}\right)^8 \right]$$

$$+ 2(1-\nu)\left[4P_2^2\left(\frac{a}{R}\right)^2 + 6P_2^2\left(\frac{a}{R}\right)^6 + 12 P_2 P_2 \left(\frac{a}{R}\right)^4 - \frac{8}{3}P_2\left(\frac{a}{R}\right)^6 - 2P_2\left(\frac{a}{R}\right)^8 + \frac{4}{9}\left(\frac{a}{R}\right)^{10} \right]$$

$$+ \left[24 P_2^2\left(\frac{a}{R}\right)^6 - 16 P_2\left(\frac{a}{R}\right)^8 + \frac{16\times 16}{3}\left(\frac{a}{R}\right)^{10} \right]$$

$$\frac{E_1}{R^3}=\left(\frac{t}{R}\right)\frac{\sigma^2}{3E}\pi\left(\frac{a}{R}\right)^2\left[\,s_2^2+2(1+\nu)\left\{(a_0^2+3s_2^2)+12\,s_2\,p_2+12\,p_2^2\right\}\right]$$

$$\frac{E_2}{R^3}=\left(\frac{t}{R}\right)\frac{\sigma^2}{3E}\pi\left(\frac{a}{R}\right)^2\ \xi^2 g^2\left[\left\{p_0^2+\frac{8}{3}\left(4-3\frac{a}{\ell}\right)p_0+34.1333-49.7778g+26.4127g^2\right\}\right.$$

$$-(1+\nu)\left\{\frac{4}{3}p_0^2+4\left(\frac{4}{3}-\frac{a}{\ell}\right)p_0+14.2222-14.2222\frac{a}{\ell}-6.2222g^2\right\}$$

$$+\left\{24\,n_2^2-16\,n_2+\frac{256}{3}\right\}$$

$$\left.+(1+\nu)\left\{8p_2^2+12\,n_2^2+24\,p_2 n_2-\frac{16}{3}p_2-4\,n_2+\frac{2}{9}\right\}\right]$$

$$\frac{E_{01}}{R^3}=\left(\frac{t}{R}\right)\frac{\sigma^2}{3E}\pi\left(\frac{a}{R}\right)^2\ 0.122100\,\frac{a^2}{R^2}$$

$$\frac{D}{R^3}=\left(\frac{t}{R}\right)\frac{\sigma^2}{3E}\pi\ \frac{a^2}{\ell^2}\left\{2(1-\nu)\ldots\right\}$$

$$\frac{E_1}{R^3}=\left(\frac{t}{R}\right)\frac{\sigma^2}{3E}\pi\left(\frac{a}{R}\right)^2\left[\,0.50587^2\,\xi^2 g^2+2.6\left\{\xi^2 g^2\frac{4}{9}(g^2-4g+4)+2\times0.50587^2\,\xi^2 g^2\right.\right.$$

$$\left.\left.-12\times0.13849\times0.50587\,\xi^2 g^2+12\times0.13849^2\,\xi^2 g^2\right\}\right]$$

$$=\left(\frac{t}{R}\right)\frac{\sigma^2}{3E}\pi\left(\frac{a}{R}\right)^2\left[\,\xi^2 g^2\left(\begin{array}{l}1.58660\\1.77777\\-0.84070\\+0.23015\end{array}\quad-1.77777g\quad+0.44244g^2\right)\right]$$

$$\frac{E_1}{R^3}=\left(\frac{t}{R}\right)\frac{\sigma^2}{3E}\pi\left(\frac{a}{R}\right)^2\left[\,\xi^2 g^2\left(0.44444g^2-1.77777g+2.75383\right)\right]$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{G^2}{2E}\pi\left(\frac{a}{R}\right)^2 \xi^2 g^2 \Big[\big\{0.35\, q_0^2 + (3.7333 - 2.80000\, g)\, q_0$$

$$+\ 15.6445 - 31.2890\, g + 34.5015\, g^2\big\}$$

$$+\big\{10.4\, p_2^2 + 39.6\, r_2^2 + 31.2\, p_2 r_2 - 6.93333\, p_2 - 21.2\, r_2 + 8.82222\big\}\Big]$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{a^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\ 0.35\big(1 - 8(\xi g) + 5.33333\, \xi g\, g\big)^2$$

$$+\ \xi g\,(3.7333 - 2.80000\, g)\big(1 - 8\xi g + 5.3333(\xi g) g\big)$$

$$+(15.6445 - 31.2890\, g + 34.5015\, g^2)\,\xi^2 g^2$$

$$+\ 10.4 \times (1.20839\, \xi g + 0.25000)^2 + 39.6 \times 0.573106\ \xi^2 g^2 - 31.2 \times 0.573106\, \xi g \times (1.20839\, \xi g + 0.25000)$$

$$+\ 6.93333\, \xi g\,(1.20839\, \xi g + 0.25000) - 21.2\, \xi g \times 0.573106\, \xi g + 8.82222\, \xi^2 g^2\Big]$$

$$\xi^2 g^2\Big[\ g^2 \begin{array}{r} 9.95555 \\ -14.93333 \\ +34.5015 \end{array} \qquad g \begin{array}{r} -29.8666 \\ +49.3111 \\ -31.2890 \end{array} \qquad \begin{array}{r} +22.40000 \\ -29.86666 \\ +15.6445 \\ +15.1862 \\ +22.5762 \\ -21.4940 \\ +8.3782 \\ -12.0862 \\ +8.8222 \end{array}$$

$$(\xi g)\Big[\, g \quad \begin{array}{l} 3.73333 \\ -2.80000 \end{array} \qquad \begin{array}{l} -56 \\ +3.73333 \\ +6.28363 \\ -4.47683 \\ +1.73333 \end{array} \Big]$$

$$+\left[\, 0.35 + 0.65 \right]$$

$$\boxed{\frac{\mathfrak{S}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\, \xi^2 g^2 \left(2.75237\, g^2 - 7.7225 + 2.95605 \right) + \xi g \left(0.93333\, g + 1.70375 \right) \Big]}$$

$$\frac{\mathfrak{P}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big\{ -1.4\ldots \; \xi g - 1.2\ldots \; \xi g \Big\}$$

$$\boxed{\frac{\mathfrak{P}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big\{ 0.89128 - 0.80000\, g \Big\}(\xi g)}$$

$$\frac{\mathfrak{C}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big\{ \xi^2 g^2 \left(29.9128\, g^2 - 20.6223\, g + 32.3143 \right) + \xi g \left(1.73333\, g + 0.81168 \right) + 0.12213\, \frac{g^2}{k^2} \Big\}$$

Let σ' be compression,

$$K^2 = \frac{0.122100\, g}{\xi(1.73333\, g + 0.81168) - \xi^2(29.9681\, g^3 - 20.6223\, g^2 + 32.3143\, g)}$$

$$\xi = \frac{1}{2}\,\frac{1.73333\, g + 0.81168}{g(29.9681\, g^2 - 20.6223\, g + 32.3143)} = \frac{1}{64}\frac{1}{\pi}\frac{L}{K}\left(\frac{a}{K}\right)^2$$

$$K^2 = \frac{0.488400\, g^2(29.9681\, g^2 - 20.6223\, g + 32.3143)}{(1.7333\, g + 0.81168)^2}$$

g = amplitude

When $g = \frac{0.89182}{0.8000} = 1.1147$

$$K^2 = \frac{0.4884 \times 1.2426 \times (37.238 - 22.988 + 32.7...)}{(2.9333...)}$$

$g = 3.1$

$$K = 0.1\,\frac{\sqrt{0.488400(30.5514)}}{0.98501} = \underline{0.3920}$$

$$\left(\frac{a}{K}\right)^2 = 32\left(\frac{L}{K}\right)\frac{\sqrt{0.488400}}{\sqrt{29.9681\, g^2 - 20.6223\, g + 32.3143}} = 404\left(\frac{L}{K}\right)$$

$\frac{L}{K} = \frac{1}{1..}$ $\quad \frac{a}{K} = 0.0636$

At $g = 0.1$

$$\xi^2 g^2 = \frac{1}{4} \frac{(1.73333g + 0.81168)^2}{(29.9681g^2 - 20.6223g + 32.3143)^2} = 0.00025985$$

$$\xi g = \frac{1}{2} \frac{1.7333g + 0.81168}{29.9681g^2 - 20.6223g + 32.3143} = \frac{1}{2} \frac{0.98501}{30.5518} = 0.016120$$

$\mathcal{E}_1 \sim 0.00025985 \times 2.58049 = \underline{0.00067054}$ + Energy outside the circular region

$\mathcal{E}_2 \sim 0.00025985 \times 27.9723 - 0.016120 \times 1.79679$

$= 0.007268 - 0.028964 = \underline{-0.021696}$ Energy — external in the region

$\mathcal{E}_3 \sim 0.17210 \frac{0.01}{0.3920^2} = \underline{+0.01195}$ Binding Energy

$\mathcal{P} \sim -0.016120 \times 0.81168 = \underline{-0.013086}$ (?) Decrease in ...

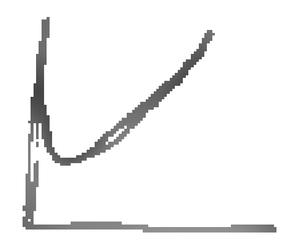
$$\frac{W_{\text{tion}}}{t} = f\,\frac{1}{4}\left(\frac{a}{R}\right)^4\left(\frac{R}{t}\right) = \frac{g}{4}\left(\frac{a}{R}\right)^2\left(\frac{R}{t}\right) = \frac{4.04}{4}\,g = 0.101 \quad (\text{Two ... all})$$

[illegible]

$$\frac{\partial\theta}{\partial g} = (\ldots)\,\frac{C^2}{2E}\,\pi\,(\ldots)^2\left\{-2(1+\nu)\,\ldots + 4\ldots\right\}$$

$$= (\ldots)\,\frac{C^2}{2E}\,\pi\,(\ldots)^2\left\{-5g \times 26 \times 0.[illegible]\,(g-2) - 2.02349\,5g\right\}$$

$$= \frac{1}{6}\,\frac{C^2}{2E}\,\pi\,(\ldots)^2\left\{1.47318 - 1.73333\,g\right\}\,5g$$



$$K^2 = \frac{0.4[illegible]\,g\,(29.9681\,g^2 - 29.6[illegible]3\,g + 22.3143)}{(9[illegible]\,g + 0.26[illegible]8)^2}$$

[illegible]

$g = 0.9$ $\qquad K = \frac{0.[illegible]\sqrt{30.[illegible]}}{[illegible]} = 0.[illegible]$

$g = 0.55$ $\qquad K = \frac{0.03[illegible]5}{0.39311}\sqrt{31.[illegible]} = 0.4[illegible]8$

$g = 0.[illegible]5$ $\qquad K = \frac{0.1048\sqrt{27.914}}{0.65[illegible]8} = 0.[illegible]0$

~~Take~~ $\cancel{2.91530\ \xi g p_2 = 2.53582\ \xi g}$

or $\cancel{\boxed{\xi g p_2 = 0.86894\ \xi g}}$

If we drop the condition of continuity of r_2

$$q_2 + 0.66667\, s_2 - 0.83333 = \xi g\{0.33333\, p_2 + [illegible]\}$$

$$q_2 \qquad - 0.83333 = \xi g\{0.33333\, p_2 - 2 r_2 - 0.33333\}$$

$$-q_2 - 0.33333\, s_2 - 0.83333 = \xi g\{0.33333\, p_2 + r_2 - 0.27222\}$$

$$q_2 \qquad - 0.750000 = \xi g\{p_2 + 3 r_2 - 0.151198\}$$

$$0.166667\, s_2 + 0 = \xi g\{-2 r_2 - [illegible]\}$$

$$-0.33333\, s_2 - [illegible] = \xi g\{[illegible]\, p_2 - [illegible]\, r_2 - 1.11111\}$$

$$-0.33333\, s_2 - 0.333[illegible] = \xi g\{1.33333\, p_2 + r_2 - 0.472103\}$$

$$s_2 + 0 = \xi g\{-3 r_2 + 1.33333\}$$

$$-s_2 - 0.5000 = \xi g\{2 p_2 + 9 r_2 - 3.33333\}$$

$$-s_2 - 1.000 = \xi g\{4 p_2 + 12 r_2 - 1.298409\}$$

$$2\xi g p_2 + 6\xi g r_2 = 2\,\xi g - 0.50000$$

$$2\xi g p_2 + 3\xi g r_2 = -2.03412\,\xi g - 0.5000$$

$$3\xi g r_2 = 4.0341[illegible]\,\xi g$$

$$\boxed{\xi g\, r_2 = 1.34487\, \xi g}$$

$$\xi g\, p_2 = \frac{1}{4}\left\{(-12.1038 - 0.03752)\,\xi g - 1.000\right\}$$

$$\boxed{\xi g\, p_2 = -3.034675\, \xi g - 0.2500}$$

$$3\, s_2 + 1.5000 = \xi g\left\{-6\, p_2 - 24\, r_2 + 5.96537\right\}$$

$$= \xi g\left\{18.2081 - 32.27688 + 5.96537\right\} + 1.5000$$

$$\boxed{s_2 = -2.70113\, \xi g}$$

$$4 q_2 + s_2 - 0.33333 = \xi g\left\{1.3333\, p_2 + 4 r_2 - 0.65513\right\}$$

$$q_2 = \frac{1}{4}\left\{2.70113 - 4.04623 + 5.37946 - 0.65513\right\}\xi g$$

$$\boxed{q_2 = 0.84481\, \xi g}$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\, 2.70113^2 + 26\Big\{\frac{4}{9}(g^2 - 4g + 4) + 2 \times 2.70113^2$$

$$- 12 \times 2.70113 \times 0.84481 + 12 \times 0.84481^2\Big\}\Big](\xi g)^2$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\, 0.44444\, g^2 - 1.77778\, g + \begin{matrix} 1.77778 \\ 7.29610 \\ 37.93972 \\ -27.38330 \\ +8.56445 \end{matrix}\Big](\xi g)^2$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\, 0.44444\, g^2 - 1.77778\, g + 38.19475\Big](\xi g)^2$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{9E}\pi\left(\frac{a}{R}\right)^2\Big[\, 0.35\,(1 - 8\,\xi g + 15 \ldots \text{[illegible]} + \xi g \ldots \text{[illegible]}$$

$$+ (34.5015\, g^2 - 31.2810\, g + 15.6445)\, \xi g^2$$

$$+ 10.4\,(3.03425\,\xi g + 2.2553)^2 + 39.5 \times 1.34447^2\, \xi g^2$$

$$- 35.02661\,\xi g\,(3.034175\,\xi g + 2.2553) - 28.51124\,(\xi g)^2 + 8.8292\,\xi g^2\Big]$$

$$(\xi g)^2\Big[\, 29.5237\, g^2 - 18.8445\, g + \begin{matrix} 22.40000 \\ -29.86666 \\ +15.6445 \\ +95.7810 \\ +71.6237 \\ -106.2944 \\ -28.5112 \\ +8.8292 \end{matrix}\Big]$$

$$\xi g \left[\begin{array}{l} 3.73333\, g \\ -7.80000 \end{array} \quad \begin{array}{r} -5.6 \\ 3.73333 \\ +15.7807 \\ -8.7567 \\ \hline \end{array} \right]$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right) \frac{\sigma^2}{2E} \pi \left(\frac{a}{R}\right)^2 \Big[(\xi g)^2 \left(29.52377 g^2 - 18.8445 g + 49.5971 \right)$$

$$+ \xi g \left(0.93333 g + 5.1573 \right) \Big]$$

$$\frac{\mathcal{P}}{R^3} = \left(\frac{t}{R}\right) \frac{\sigma^2}{2E} \pi \left(\frac{a}{R}\right)^2 \left\{ - \xi g \times 2.6 \times 0.66667 (g-2) - 10.8045 \xi g \right\}$$

$$= \left(\frac{t}{R}\right) \frac{\sigma^2}{2E} \pi \left(\frac{a}{R}\right)^2 (\xi g) \left\{ - 1.73333 g - 7.3378 \right\}$$

$$\left(\frac{w}{R}\right)_0 = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2}$$

$$\left(\frac{w}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2} - f\left(\frac{1}{2}\frac{a^2}{R^2}\right)\left\{J_0\left(\beta \frac{r}{a}\right) + \gamma\right\}\frac{1}{\delta}$$

$\beta = 3.8317$

$\gamma = 0.4028$

$\delta = 1.4028$

$$\frac{1}{R}\frac{\partial w}{\partial r} = -r\left(\frac{1}{R^2}\right)\sin^2\theta - \frac{f}{\delta}\frac{1}{2}\frac{a^2}{R^2}\frac{\beta}{a} J_0'\left(\beta\frac{r}{a}\right)$$

$$\frac{1}{R}\frac{\partial w_0}{\partial r} = -r\left(\frac{1}{R^2}\right)\sin^2\theta$$

$$\frac{1}{R}\frac{\partial^2 w}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta - \frac{f}{\delta}\frac{1}{2}\frac{a^2}{R^2}\frac{\beta^2}{a^2} J_0''\left(\beta\frac{r}{a}\right)$$

$$\frac{1}{R}\frac{\partial^2 w_0}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\}$$

$$= \frac{1}{R^2}(\sin^2\theta)^2 - \frac{1}{R^2}\left[\sin^2\theta + \frac{1}{r}\frac{f}{\delta}\frac{a\beta}{2} J_0'\left(\beta\frac{r}{a}\right)\right]\left[\sin^2\theta + \frac{1}{2}\frac{f}{\delta}\beta^2 J_0''\left(\beta\frac{r}{a}\right)\right]$$

$$= -\frac{1}{R^2}\left[\frac{1}{2}\frac{f}{\delta}\beta^2\sin^2\theta\left\{J_0'' + \frac{1}{\left(\beta\frac{r}{a}\right)}J_0'\right\} + \frac{1}{4}\frac{f^2}{\delta^2}\frac{a\beta^3}{r}J_0' J_0''\right]$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right\}$$

$$= -\frac{1}{R^2}\cos 2\theta \cdot \frac{1}{2}\frac{f}{\delta}\beta^2 J_0''$$

$$\nabla^4\phi = \frac{E}{R^2}\left[J_0 \frac{1}{2}\frac{f}{\delta}\beta^2 \sin^2\theta - \frac{1}{4}\frac{f^2}{\delta^2}\frac{a\beta^3}{r} J_0' J_0'' - \frac{1}{2}\frac{f}{\delta}\beta^2 J_0'' \cos 2\theta \right] \quad \underline{\underline{362}}$$

$$= \frac{E}{R^2}\left[J_0 \frac{1}{4}\frac{f}{\delta}\beta^2 (1-\cos 2\theta) - \frac{1}{4}\frac{f^2}{\delta^2}\frac{a\beta^3}{r} J_0' J_0'' - \frac{1}{2}\frac{f}{\delta}\beta^2 J_0'' \cos 2\theta \right]$$

$$= \frac{1}{4}\frac{f}{\delta}\beta^2 \frac{E}{R^2}\left[\left\{ J_0 - \beta^2 \frac{f}{\delta} \frac{J_0' J_0''}{(\beta \frac{r}{a})} \right\} - \cos 2\theta \left\{ J_0 + 2J_0'' \right\} \right]$$

$$= \frac{1}{4}\frac{gE}{R^2}\left[\left\{ J_0 - g\frac{J_0' J_0''}{z} \right\} - \cos 2\theta \left(J_0 + 2J_0'' \right) \right\}$$

where $\quad g = \frac{f}{\delta}\beta^2 \quad , \quad z = (\beta \frac{r}{a})$

$$\frac{J_0' J_0''}{z} = \frac{J_1 J_1'}{z} = \frac{J_1^2}{z^2} - \frac{J_1 J_2}{z}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2)! \, (\frac{1}{2}z)^{2n}}{4 (n!)(n+1)! (n+1)! (n+2)} - \sum_{n=0}^{\infty} \frac{(-1)^n (2n+3)! \, (\frac{1}{2}z)^{2n+2}}{2 (n!)(n+1)! (n+2)! (n+2)}$$

$$= \frac{1}{4} - \frac{1}{4}\left(\frac{z}{2}\right)^2 + \frac{5}{48}\left(\frac{z}{2}\right)^4 - \frac{7}{368}\left(\frac{z}{2}\right)^6 + \frac{7}{1920}\left(\frac{z}{2}\right)^8 - \frac{11}{28600}\left(\frac{z}{2}\right)^{10}$$

$$+ \frac{143}{4838400}\left(\frac{z}{2}\right)^{12} - \cdots$$

$$- \left\{ \frac{1}{4}\left(\frac{z}{2}\right)^2 - \frac{5}{24}\left(\frac{z}{2}\right)^4 + \frac{7}{96}\left(\frac{z}{2}\right)^6 - \frac{7}{480}\left(\frac{z}{2}\right)^8 + \frac{11}{5760}\left(\frac{z}{2}\right)^{10} + \frac{143}{806400}\left(\frac{z}{2}\right)^{12} \right\}$$

$$\frac{J_0 J_0''}{z} = \frac{1}{4} - \frac{1}{2}\left(\frac{z}{2}\right)^2 + \frac{5}{16}\left(\frac{z}{2}\right)^4 - \frac{7}{72}\left(\frac{z}{2}\right)^6 + \frac{7}{384}\left(\frac{z}{2}\right)^8 - \frac{11}{4800}\left(\frac{z}{2}\right)^{10} + \frac{143}{691200}\left(\frac{z}{2}\right)^{12} - \cdots$$

The particular integral for this term is

$$\frac{\phi_1}{\lambda^2} = -\frac{1}{4}\left(\frac{a}{R}\right)^4\left(\frac{f}{\delta}\right)^2 E\left\{\frac{1}{16}\left(\frac{z}{2}\right)^4 - \frac{1}{72}\left(\frac{z}{2}\right)^6 + \frac{5}{2304}\left(\frac{z}{2}\right)^8 - \frac{7}{28800}\left(\frac{z}{2}\right)^{10} + \frac{7}{345600}\left(\frac{z}{2}\right)^{12} - \frac{11}{41211600}\left(\frac{z}{2}\right)^{14} + \frac{143}{691200 \times 56 \times 64}\left(\frac{z}{2}\right)^{16} - \cdots\right\}$$

The particular integral for the term $\frac{1}{4}\frac{gE}{\lambda^2} J_0$ is

$$\frac{\phi_2}{\lambda^2} = \frac{1}{4}\left(\frac{a}{R}\right)^4\left(\frac{f}{\delta}\right)^2 \frac{E}{\beta^2} J_0$$

The particular integral for the term $-\frac{1}{4}\frac{gE}{\lambda^2}\cos 2\theta\, J_2$ is

$$\frac{\phi_3}{\lambda^2} = -\frac{1}{4}\left(\frac{a}{R}\right)^4\left(\frac{f}{\delta}\right)^2 \frac{E}{\beta^2}\cos 2\theta\, J_2$$

Then the total particular integral is

$$\frac{\phi_1+\phi_2+\phi_3}{\lambda^2} = \frac{1}{4}\left(\frac{a}{R}\right)^4 \frac{1}{\beta^4}\, gE\left[\left\{J_0 - \frac{\beta^2}{[illegible]}\left[\frac{1}{4}\left(\frac{z}{2}\right)^4 - \frac{1}{18}\left(\frac{z}{2}\right)^6 + \frac{5}{576}\left(\frac{z}{2}\right)^8 - \frac{7}{7200}\left(\frac{z}{2}\right)^{10} + \frac{7}{86400}\left(\frac{z}{2}\right)^{12} - \frac{11}{2116800}\left(\frac{z}{2}\right)^{14} + \frac{143}{691200 \times 56 \times 64}\left(\frac{z}{2}\right)^{16} - \cdots\right]\right\} - J_2\cos 2\theta\right\} = \frac{\Phi}{\lambda^2}$$

The stresses due to this particular internal are:

$$\frac{1}{2}\frac{\partial\Phi}{\partial r} = \left(\frac{\beta}{a}\right)^2 \frac{1}{z}\frac{\partial\Phi}{\partial z} = \frac{1}{4}\left(\frac{a}{\beta}\right)^2 \frac{GE}{\beta^2}\left[\frac{J_0'}{z} - \frac{a}{4}\left[\frac{1}{4}\left(\frac{z}{2}\right)^2 - \frac{1}{12}\left(\frac{z}{2}\right)^4 + \frac{[illegible]}{288}\left(\frac{z}{2}\right)^6 \right.\right.$$

$$\left. - \frac{7}{2880}\left(\frac{z}{2}\right)^8 + \frac{7}{28800}\left(\frac{z}{2}\right)^{10} - \frac{11}{604800}\left(\frac{z}{2}\right)^{12} + \frac{143}{135475200}\left(\frac{z}{2}\right)^{14} - \cdots\right]$$

$$\left. - \frac{J_2'}{z}\cos 2\theta\right\}$$

$$\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = \left(\frac{\beta}{a}\right)^2 \frac{1}{z^2}\frac{\partial^2\Phi}{\partial\theta^2} = \frac{1}{4}\left(\frac{a}{\beta}\right)^2 \frac{GE}{\beta^2}\left[+\frac{J_2}{z^2}\,4\cos 2\theta\right]$$

$$\widehat{rr}_1 = \frac{1}{4}\left(\frac{a}{\beta}\right)^2 \frac{GE}{\beta^2}\left[\frac{J_0'}{z} - \frac{a}{4}\left\{\frac{1}{4}\left(\frac{z}{2}\right)^2 - [illegible]\left(\frac{z}{2}\right)^4 + \frac{[illegible]}{288}\left(\frac{z}{2}\right)^6 - \frac{7}{2880}\left(\frac{z}{2}\right)^8 + \frac{7}{28800}\left(\frac{z}{2}\right)^{10}\right.\right.$$

$$\left.\left. - \frac{11}{604800}\left(\frac{z}{2}\right)^{12} + \frac{143}{135475200}\left(\frac{z}{2}\right)^{14} - \cdots\right\} - \cos 2\theta\left(\frac{J_2'}{z} - \frac{4J_2}{z^2}\right)\right]$$

$$\widehat{\theta\theta}_1 = \frac{1}{4}\left(\frac{a}{\beta}\right)^2 \frac{GE}{\beta^2}\left\{J_0'' - \frac{a}{4}\left[\frac{3}{4}\left(\frac{z}{2}\right)^2 - \frac{5}{12}\left(\frac{z}{2}\right)^4 + \frac{[illegible]}{288}\left(\frac{z}{2}\right)^6 - \frac{63}{2880}\left(\frac{z}{2}\right)^8 + \frac{77}{28800}\left(\frac{z}{2}\right)^{10}\right.\right.$$

$$\left.\left. - \frac{143}{604800}\left(\frac{z}{2}\right)^{12} + \frac{2145}{135475200}\left(\frac{z}{2}\right)^{14} - \cdots\right] - J_2''\cos 2\theta\right\}$$

$$\widehat{r\theta}_1 = \frac{1}{4}\left(\frac{a}{\beta}\right)^2 \frac{GE}{\beta^2}\left[2\sin 2\theta\left(\frac{J_2'}{z} - \frac{J_2}{z^2}\right)\right]$$

$$\frac{1}{E}(\widehat{rr} - \nu\,\widehat{\theta\theta}) = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{g}{\beta^2}\left[\left(\frac{J_0'}{z} - \nu J_0''\right) - g\left\{\frac{(1-3\nu)}{16}\left(\frac{z}{2}\right)^2 - \frac{(1-5\nu)}{48}\left(\frac{z}{2}\right)^4\right.\right.$$

$$+ \frac{5(1-7\nu)}{288\times 4}\left(\frac{z}{2}\right)^6 - \frac{7(1-9\nu)}{2880\times 4}\left(\frac{z}{2}\right)^8 + \frac{7(1-11\nu)}{4038600}\left(\frac{z}{2}\right)^{10} - \frac{11(1-13\nu)}{604450\times 4}\left(\frac{z}{2}\right)^{12}$$

$$\left.\left. + \frac{143(1-15\nu)}{4\times 135475200}\left(\frac{z}{2}\right)^{14} - \right\} - \cos 2\theta\left\{\frac{J_2'}{z} - \frac{4J_2}{z^2} - \nu J_2''\right\}\right]$$

$$\frac{1}{2}\left\{\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2\right\} = \frac{1}{2}\left[\frac{1}{2}\left(\frac{a}{R}\right)\beta\frac{g}{\delta}J_0'\left\{\frac{1}{2}\left(\frac{a}{R}\right)\beta\frac{g}{\delta}J_0' + 2\left(\frac{a}{R}\right)\sin^2\theta\right\}\right]$$

$$= \frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\left[\left(z J_0' + \frac{g J_0'^2}{2}\right) - z J_0' \cos 2\theta\right]$$

$$J_0'^2 = J_1^2 = \sum_n \frac{(-1)^n (2n+2)!\left(\frac{z}{2}\right)^{2n+2}}{n!\,(n+2)!\,(n+1)!\,(n+1)!}$$

$$= \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^4 + \frac{5}{12}\left(\frac{z}{2}\right)^6 - \frac{7}{72}\left(\frac{z}{2}\right)^8 +$$

Therefore

$$\frac{\partial u}{\partial r} = \frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\left[\left\{\left(\frac{J_0'}{z} - z J_0'\right) - \nu J_0''\right\} - g\left\{\frac{3(3-\nu)}{16}\left(\frac{z}{2}\right)^2 - \frac{5(5-\nu)}{48}\left(\frac{z}{2}\right)^4\right.\right.$$

$$+ \frac{5\times 7(7-\nu)}{1152}\left(\frac{z}{2}\right)^6 - \frac{7\times 9(9-\nu)}{11520}\left(\frac{z}{2}\right)^8 + \frac{7\times 11(11-\nu)}{115200}\left(\frac{z}{2}\right)^{10} - \frac{11\times 13(13-\nu)}{2419200}\left(\frac{z}{2}\right)^{12}$$

$$\left.\left. + \frac{143\times 15(15-\nu)}{541900800}\left(\frac{z}{2}\right)^{14} - - \right\} - \cos 2\theta\left(\frac{J_2'}{z} - \frac{4J_2}{z^2} - \nu J_2'' - z J_0'\right)\right]$$

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$$\frac{u_1}{R}=\frac{1}{4}\left(\frac{a}{R}\right)^3\frac{1}{\beta^3}g\Big[\Big\{(1\quad z J_0''\quad)-\nu J_0'\Big\}-g\Big\{\frac{(3-\nu)}{8}\left(\frac{z}{2}\right)^3-\frac{(5-\nu)}{24}\left(\frac{z}{2}\right)^5$$

$$+\frac{5(7-\nu)}{576}\left(\frac{z}{2}\right)^7-\frac{7(9-\nu)}{5760}\left(\frac{z}{2}\right)^9+\frac{7(11-\nu)}{57600}\left(\frac{z}{2}\right)^{11}-\frac{11(13-\nu)}{1209600}\left(\frac{z}{2}\right)^{13}$$

$$+\frac{143(15-\nu)}{270950400}\left(\frac{z}{2}\right)^{15}-\cdots\Big\}-\cos 3\theta\Big\{-4J_2'\quad J_2'-zJ_1'-J_0'\Big\}$$

$$\int\left(\frac{J_2'}{z}-\frac{4J_2}{z^2}\quad zJ_0'\right)dz=\int\Big\{\frac{J_2'}{z}-\left(J_2'+\frac{J_2'}{z}+J_2\right)-zJ_0'\Big\}dz$$

$$=-J_2'-\int(J_2+zJ_0')\,dz$$

$$=-J_2'-\int\Big\{\frac{2J_1}{z}-J_0(z)+zJ_0'\Big\}dz=-J_2'+\int\left(\frac{J_0'}{z}-zJ_0\right)dz$$

$$+\int\left(\frac{J_0'}{z}+J_0\right)dz=\quad-zJ_0''+\int\left(\frac{J_0'}{z}-J_0\right)dz=-J_2'+\ldots-J_0'$$

But $\quad J_0=J_2-2J_0'$

$$\frac{J_0'}{z}=-J_2+J_0''$$

$$J_0+\frac{J_0'}{z}=-J_0'$$

$$\frac{u}{R}=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{1}{\beta^2}\Big[\Big\{J_0'-\nu\frac{J_0}{z}\Big\}-g\Big\{\frac{3-\nu}{16}\left(\frac{z}{2}\right)^2-\frac{(5-\nu)}{48}\left(\frac{z}{2}\right)^4+\cdots\Big\}$$

$$+\cos\theta\Big\{\frac{J_2'}{z}+J_1'+\frac{J_2'}{z}+\nu\frac{J_2'}{z}\Big\}\Big]$$

$$-\frac{4}{2}\frac{1}{E}(\widehat{\theta\theta}_1-\widehat{rr}_1) = \frac{1}{4}\left(\frac{a}{R}\right)^2\frac{2}{\beta^2}\left[\cos 2\theta\left\{-J_2''+\frac{\nu J_2'}{z}-\frac{4\nu J_2}{z^2}-\frac{J_2'}{z}-J_1'-\frac{J_1}{z}-\frac{\nu J_2'}{z}\right\}\right]$$

$$=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{2}{\beta^2}\left[\cos 2\theta\left\{-J_2''-\frac{J_2'}{z}-J_1'+\frac{J_1}{z}-\frac{4\nu J_2}{z^2}\right\}\right]$$

$$\cancel{=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{2}{\beta^2}\left[\cos 2\theta\left\{J_2-\frac{4(1+\nu)J_2}{z^2}+\frac{J_2}{z}-J_1'\right\}\right]}$$

$$\boxed{\frac{V}{R}=\frac{1}{4}\left(\frac{a}{R}\right)^3\frac{2}{\beta^3}\left[\frac{\sin 2\theta}{2}\left\{-2J_2''-J_2'-2J_1'+J_1-\frac{4\nu J_2}{z}\right\}\right]}$$

The total stress components can be [illegible]

$$\widehat{rr}=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{2E}{\beta^2}\left[\frac{1}{2}Q_0-\frac{z}{2}-\frac{2}{4}\left\{\frac{4}{4}\left(\frac{z}{2}\right)^2-\frac{1}{12}\left(\frac{z}{2}\right)^4+\frac{5}{288}\left(\frac{z}{2}\right)^6-\frac{7}{2880}\left(\frac{z}{2}\right)^8+\dots\right.\right.$$

$$\left.\left.-\frac{11}{604800}\left(\frac{z}{2}\right)^{12}+\frac{143}{1354752\dots}\left(\frac{z}{2}\right)^{14}-\dots\right\}-\cos 2\theta\left\{2P_2+\frac{J_2}{z}-\frac{6J_2}{z^2}\right\}\right]$$

$$\widehat{\theta\theta}=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{2E}{\beta^2}\left[\frac{1}{2}Q_0-\frac{1}{2}(J_0-J_2)-\frac{2}{4}\left\{\frac{3}{4}\left(\frac{z}{2}\right)^2-\frac{5}{12}\left(\frac{z}{2}\right)^4+\frac{35}{288}\left(\frac{z}{2}\right)^6-\frac{63}{2880}\left(\frac{z}{2}\right)^8\right.\right.$$

$$\left.+\frac{77}{268800}\left(\frac{z}{2}\right)^{10}-\frac{143}{604800}\left(\frac{z}{2}\right)^{12}+\frac{2145}{1354752200}\left(\frac{z}{2}\right)^{14}-\dots\right\}+\cos 2\theta\left\{2P_2+12P_2z^2\right.$$

$$\left.\left.-\left(\frac{6}{z^2}-1\right)J_2+\frac{J_1}{z}\right\}\right]$$

$$\widehat{r\theta}=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{2E}{\beta^2}\left[\sin 2\theta\left\{2P_2+6P_2z^2-\frac{6J_2}{z^2}+\frac{3J_1}{z}\right\}\right]$$

The [illegible] deflection is

$$\frac{u}{R} = \frac{1}{4}\left(\frac{a}{R}\right)^3 \frac{q}{\beta^3}\Big[\frac{(1-\nu)}{2} D_0 z + J_1 - zJ_0 + \nu J_1 - \frac{q}{4}\Big\{ \frac{(3-\nu)}{2}\left(\frac{z}{2}\right)^3 - \frac{(5-\nu)}{6}\left(\frac{z}{2}\right)^5$$

$$+ \frac{5(7-\nu)}{144}\left(\frac{z}{2}\right)^7 - \frac{7(9-\nu)}{1440}\left(\frac{z}{2}\right)^9 + \frac{7(11-\nu)}{14400}\left(\frac{z}{2}\right) - \frac{11(13-\nu)}{[illegible]}\left(\frac{z}{2}\right)^{13}$$

$$+ \frac{143(15-\nu)}{67737600}\left(\frac{z}{2}\right)^{15} - \dots \Big\} - \cos 2\theta \Big\{ 2(1+\nu) P_2 z + 4\nu P_2 z^3 + J_1 - zJ_1'$$

$$- (1+\nu) J_2' \Big\} \Big]$$

$$\frac{v}{R} = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{q}{\beta^2}\Big[\cos 2\theta \Big\{ [illegible] P_2 z + 6(1+\nu) P_2 z^3 - \frac{[illegible]}{2} - \frac{[illegible]}{2} - \frac{[illegible]}{2}$$

$$+ \frac{1}{z} J_1 - \frac{2\nu J_2}{z} \Big\} \Big]$$

At [illegible] = [illegible] $z = [illegible]$ [illegible] = 1.6159

$$n_a = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{qE}{\beta^2}\Big[\frac{1}{2} D_0 - \frac{q}{4}\Big\{ 0.25 \times 1.6159^2 - 0.083333 \times 1.6159^4 + 0.00173611 \times 1.6159^6$$

$$- 0.0002430556 \times 1.6159^8 + 0.0000243056 \times 1.6159^{10} - 0.00001[illegible] \times 1.6159^{12}$$

$$+ 0.00000105557 \times 1.6159^{14} - \dots \Big\} - \cos 2\theta \Big\{ 3P_2 - 6\,\frac{0.4025}{3.8314^2} \Big\} \Big]$$

$$= \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{qE}{\beta^2}\Big[\frac{1}{2} D_0 - \frac{q}{4}\Big\{ 0.25 \times 2.611 - 0.083333 \times 6.8178 + 0.001736 \times 17.80[illegible]$$

$$- 0.0002430556 \times 46.828 + 0.0000243056 \times 121.37 - 0.0000[illegible] \times 316.91$$

$$+ 0.00000105557 \times 822.38 - \dots \Big\} - [illegible] \Big\{ 3P_2 - 0.1645 \Big\} \Big]$$

+0.65278	+1.95834
−0.56815	−2.84025
+0.03091	+0.21637
−0.01130	−0.10170
+0.00295	+0.03245
−0.00576	−0.02487
+0.00087	+0.01305
0.100	−0.824

$$\lambda_1 = \frac{1}{4}\left(\frac{a}{R_1}\right)^2 \frac{\rho E}{\beta^2}\left[\frac{4}{3}Q_1 - 0.02509\, q - \cos 2\theta\left(2P_2 - 0.1645\right)\right]$$

$$\theta_2 = \frac{1}{4}\left(\frac{a}{R_1}\right)^2 \frac{\rho E}{\beta^2}\left[\frac{4}{3}Q_1 + 0.4027 + 0.2469\, q + \cos 2\theta\left\{2P_2 + 17.618\, R_2 + 0.2310\right\}\right]$$

$$\lambda_2 = \frac{1}{4}\left(\frac{a}{R_1}\right)^2 \frac{\rho E}{\beta^2}\left[\sin 2\theta\left\{2P_2 + 11.09\, R_2 - 0.1145\right\}\right]$$

[illegible]

$$= \frac{1}{4}\left(\frac{a}{R_1}\right)^2 \frac{\rho E}{\beta^2}\left[- \ldots \; 2 \ldots \; + 3.2221\, R_2 + 0.4027 + \ldots 0.0051 B\right]$$

$$\frac{v}{R_a} = \frac{1}{4}\left(\frac{a}{R_1}\right)^2 \frac{\rho E}{\beta^2}\left[\sin 2\theta\left\{2(1+\nu)P_2 + 88.0914(1+\nu)R_2 + 0.3428 - 0.0544\,\nu\right\}\right]$$

Put $\frac{1}{4}\left(\frac{a}{R_1}\right)^2 \frac{\rho E}{\beta^2} = \eta$

Then the stress + displacement conditions give

$$\frac{1}{2} + r_0 = \eta \frac{g}{l} \left\{ \frac{1}{2} Q_0 - 0.0250 \frac{g}{l} \right\} \qquad (1)$$

$$\frac{1}{2} - r_0 = \eta \frac{g}{l} \left\{ \frac{1}{2} Q_0 + 0.4027 + 0.206 \frac{g}{l} \right\} \qquad (2)$$

$$3) \quad \frac{1}{2} - 6 q_{/2} - 4 s_2 = \eta \frac{g}{l} \left\{ 0.1645 - 2 P_2 \right\}$$

$$4) \quad 6 q_{/2} - \frac{1}{2} = \eta \frac{g}{l} \left\{ 2 P_2 + 126.18 P_2 + 0.2 [illegible] \right\}$$

$$5) \quad \frac{1}{2} + 6 q_{/2} + 2 s_2 = \eta \frac{g}{l} \left\{ 0.1645 - [illegible] - [illegible] P_2 \right\}$$

$$\frac{1}{2}(1+\lambda) + 2(1+\lambda) q_{/2} + 4 s_2 = \eta \frac{g}{l} \left\{ - [illegible] - [illegible] P_2 - [illegible] \right\} \quad [illegible]$$

$$2(1+\lambda) q_{/2} - \frac{1}{2}(1+\lambda) = \eta \frac{g}{l} \left\{ 2(1+\lambda) [illegible] \right\}$$

From (1) + (2)

$$1 = \eta \frac{g}{l} \left\{ Q_0 + 0.4027 + 0.181 \frac{g}{l} \right\}$$

$$\boxed{Q_0 = \frac{1}{\eta \frac{g}{l}} - \left(0.4027 + 0.181 \frac{g}{l} \right)}$$

$$r_0 = \eta \frac{g}{l} \left\{ - \left(0.20135 + [illegible] \frac{g}{l} \right) - [illegible] \right\} = - \eta \frac{g}{l} \left\{ 0.20135 + 0.155 \frac{g}{l} \right\}$$

$$\boxed{r_0 = - \eta \frac{g}{l} \left\{ 0.20 + [illegible] \frac{g}{l} \right\}}$$

$$q_2 + 0.6667\,S_2 - 0.08333 = \frac{\eta g}{l}\left\{0.3333\,P_2 - 0.02742\right\}$$

$$q_2 + \quad 0 - 0.08333 = \frac{\eta g}{l}\left\{0.3333\,P_2 + 29.36\,R_2 + 0.03967\right\}$$

$$q_2 + 0.3333\,S_2 + 0.08333 = \frac{\eta g}{l}\left\{-0.3333\,P_2 - 14.68\,R_2 + 0.02742\right\}$$

$$q_2 + 1.5385\,S_2 + 0.2500 = \frac{\eta g}{l}\left\{-P_2 - 6.7260\,R_2 - 0.1823\right\}$$

$$q_2 + \quad 0 - 0.2500 = \frac{\eta g}{l}\left\{P_2 + 44.0457\,R_2 + 0.1275\right\}$$

$$0.6667\,S_2 + 0 = \frac{\eta g}{l}\left\{-29.36\,R_2 - 0.06709\right\}$$

$$0.3333\,S_2 + 0.1667 = \frac{\eta g}{l}\left\{-0.6667\,P_2 - 44.04\,R_2 - 0.01225\right\}$$

$$1.2052\,S_2 + 0.1667 = \frac{\eta g}{l}\left\{-0.6667\,P_2 + 7.904\,R_2 - 0.2097\right\}$$

$$1.5385\,S_2 + 0.5000 = [illegible]\left\{-2\,P_2 - 50.83\,R_2 - 0.3098\right\}$$

$$S_2 + \quad 0 = \frac{\eta g}{l}\left\{-44.04\,R_2 - 0.1006\right\}$$

$$S_2 + 0.5000 = \frac{\eta g}{l}\left\{-2\,P_2 - 132.12\,R_2 - 0.03675\right\}$$

$$S_2 + 0.1383 = \frac{\eta g}{l}\left\{-0.5552\,P_2 + 6.556\,R_2 - 0.1740\right\}$$

$$S_2 + 0.3250 = \frac{\eta g}{l}\left\{-1.3\,P_2 - 33.038\,R_2 - 0.2014\right\}$$

$$0.5000 = \frac{\eta g}{l}\left\{-2\,P_2 - 88.08\,R_2 + 0.06385\right\}$$

$$0.3617 = \frac{\eta g}{l}\left\{-1.4458\,P_2 - 138.68\,R_2 + 0.1372\right\}$$

$$0.1867 = \frac{\eta g}{l}\left\{-0.7465\,P_2 - 37.526\,R_2 - 0.0274\right\}$$

M. A. Biot
ERNEST KEMPTON ADAMS FUND
FOR PHYSICAL RESEARCH
PUPIN PHYSICS LABORATORIES
COLUMBIA UNIVERSITY
NEW YORK CITY

[illegible]

Dr. H. S. Tsien
Guggenheim Laboratory
California Institute of Technology
Pasadena
California

For the circular region,

$$\left(\frac{w}{R}\right)_0 = \frac{1}{2}\left(\frac{a}{R}\right)^2 \left\{ 1 - \left(\frac{r}{a}\right)^2 \sin^2\theta \right\}$$

$$\frac{w}{R}\Big) = \frac{1}{2}\left(\frac{a}{R}\right)^2 \left\{ 1 - \left(\frac{r}{a}\right)^2 \sin^2\theta - f_1\left[1 - \left(\frac{r}{a}\right)^2\right]^2 - f_2\left[1 - \left(\frac{r}{a}\right)^3\right]^2 - f_3\left[1 - \left(\frac{r}{a}\right)^4\right]^2 \right\}$$

where f_1, f_2, f_3, are amplitudes.

$$\frac{1}{r}\frac{\partial w}{\partial r} = \frac{1}{2}\frac{1}{R}\left\{ - 2\sin^2\theta + 4f_1\left[1 - \left(\frac{r}{a}\right)^2\right] + 6f_2\left[1 - \left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 8f_3\left[1 - \left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2 \right\}$$

$$\frac{1}{r}\left(\frac{\partial w}{\partial r}\right)_0 = \frac{1}{2}\frac{1}{R}\left\{ -2 \sin^2\theta \right\}$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{1}{2}\frac{1}{R}\left\{ -2\sin^2\theta + 4f_1\left[1 - 3\left(\frac{r}{a}\right)^2\right] + 6f_2\left[2 - 5\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 8f_3\left[3 - 7\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2 \right\}$$

$$\frac{\partial^2 w_0}{\partial r^2} = \frac{1}{2}\frac{1}{R}\left\{ -2\sin^2\theta \right\}$$

$$\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2} = \frac{1}{R}\left\{ -\cos 2\theta \right\}$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}-\frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\}$$

$$=\frac{1}{R^2}(\cos^2\theta)^2-\frac{1}{R^2}\left\{\sin^2\theta-2f_1\left[1-\left(\tfrac{r}{a}\right)^2\right]-3f_2\left[1-\left(\tfrac{r}{a}\right)^3\right]\left(\tfrac{r}{a}\right)-4f_3\left[1-\left(\tfrac{r}{a}\right)^4\right]\left(\tfrac{r}{a}\right)^2\right\}\left\{\sin^2\theta-2f_1\left[1-3\left(\tfrac{r}{a}\right)^2\right]-3f_2\left[2-5\left(\tfrac{r}{a}\right)^3\right]\left(\tfrac{r}{a}\right)-4f_3\left[3-7\left(\tfrac{r}{a}\right)^4\right]\left(\tfrac{r}{a}\right)^2\right\}$$

$$=\frac{1}{R^2}\left\{2f_1\left[1-2\left(\tfrac{r}{a}\right)^2\right]+3f_2\left[1.5-3\left(\tfrac{r}{a}\right)^3\right]\left(\tfrac{r}{a}\right)+4f_3\left[2-4\left(\tfrac{r}{a}\right)^4\right]\left(\tfrac{r}{a}\right)^2\right\}\overline{(1-\cos 2\theta)}$$

$$\begin{aligned}-\frac{1}{R^2}\Big\{&4f_1^2\left[1-4\left(\tfrac{r}{a}\right)^2+3\left(\tfrac{r}{a}\right)^4\right]+6f_1f_2\left[1-3\left(\tfrac{r}{a}\right)^2-\left(\tfrac{r}{a}\right)^3+3\left(\tfrac{r}{a}\right)^5\right]\left(\tfrac{r}{a}\right)+8f_1f_3\left[1-3\left(\tfrac{r}{a}\right)^2-\left(\tfrac{r}{a}\right)^4+3\left(\tfrac{r}{a}\right)^6\right]\left(\tfrac{r}{a}\right)^2\\&+6f_1f_2\left[2-2\left(\tfrac{r}{a}\right)^2-5\left(\tfrac{r}{a}\right)^3+5\left(\tfrac{r}{a}\right)^5\right]\left(\tfrac{r}{a}\right)+8f_1f_3\left[3-3\left(\tfrac{r}{a}\right)^2-7\left(\tfrac{r}{a}\right)^4+7\left(\tfrac{r}{a}\right)^6\right]\left(\tfrac{r}{a}\right)^2\\&+9f_2^2\left[2-7\left(\tfrac{r}{a}\right)^3+5\left(\tfrac{r}{a}\right)^6\right]\left(\tfrac{r}{a}\right)^2+12f_2f_3\left[2-5\left(\tfrac{r}{a}\right)^3-2\left(\tfrac{r}{a}\right)^4+5\left(\tfrac{r}{a}\right)^7\right]\left(\tfrac{r}{a}\right)^3\\&+12f_2f_3\left[3-3\left(\tfrac{r}{a}\right)^3-7\left(\tfrac{r}{a}\right)^4+7\left(\tfrac{r}{a}\right)^7\right]\left(\tfrac{r}{a}\right)^3+16f_3^2\left[3-10\left(\tfrac{r}{a}\right)^4+7\left(\tfrac{r}{a}\right)^8\right]\left(\tfrac{r}{a}\right)^4\Big\}\end{aligned}$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial \theta^2}-\frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right\}$$

$$=\frac{1}{R^2}\cos 2\theta\left\{2f_1\left[1-3\left(\tfrac{r}{a}\right)^2\right]+3f_2\left[2-5\left(\tfrac{r}{a}\right)^3\right]\left(\tfrac{r}{a}\right)+4f_3\left[3-7\left(\tfrac{r}{a}\right)^4\right]\left(\tfrac{r}{a}\right)^2\right\}$$

The terms for the particular integral is then, <u>multiplied by</u> R^2,

$$2f_1\left[1-2\left(\tfrac{r}{a}\right)^2\right]+3f_2\left[1.5-3\left(\tfrac{r}{a}\right)^3\right]\left(\tfrac{r}{a}\right)+4f_3\left[2-4\left(\tfrac{r}{a}\right)^4\right]\left(\tfrac{r}{a}\right)^2$$

$$-4f_1^2\left[1-4\left(\tfrac{r}{a}\right)^2+3\left(\tfrac{r}{a}\right)^4\right]-6f_1f_2\left[3-5\left(\tfrac{r}{a}\right)^2-6\left(\tfrac{r}{a}\right)^3+8\left(\tfrac{r}{a}\right)^5\right]\left(\tfrac{r}{a}\right)-8f_1f_3\left[4-6\left(\tfrac{r}{a}\right)^2-8\left(\tfrac{r}{a}\right)^4+10\left(\tfrac{r}{a}\right)^6\right]\left(\tfrac{r}{a}\right)^2$$

$$-9f_2^2\left[2-7\left(\tfrac{r}{a}\right)^3+5\left(\tfrac{r}{a}\right)^6\right]\left(\tfrac{r}{a}\right)^2-12f_2f_3\left[5-8\left(\tfrac{r}{a}\right)^3-9\left(\tfrac{r}{a}\right)^4+12\left(\tfrac{r}{a}\right)^7\right]\left(\tfrac{r}{a}\right)^3-16f_3^2\left[3-10\left(\tfrac{r}{a}\right)^4+7\left(\tfrac{r}{a}\right)^8\right]\left(\tfrac{r}{a}\right)^4$$

$$-\cos 2\theta\left\{2f_1\left(\tfrac{r}{a}\right)^2-3f_2\left[0.5-2\left(\tfrac{r}{a}\right)^3\right]\left(\tfrac{r}{a}\right)-4f_3\left[1-3\left(\tfrac{r}{a}\right)^4\right]\left(\tfrac{r}{a}\right)^2\right\}$$

$$=2f_1(1-2f_1)+(4.5f_2-18f_1f_2)\left(\tfrac{r}{a}\right)+(-4f_1+8f_3+16f_1^2-32f_1f_3-18f_2^2)\left(\tfrac{r}{a}\right)^2$$

$$+(30f_1f_2-60f_2f_3)\left(\tfrac{r}{a}\right)^3+(-9f_2-12f_1^2+36f_1f_2+48f_1f_3-48f_3^2)\left(\tfrac{r}{a}\right)^4$$

$$+(63f_2^2)\left(\tfrac{r}{a}\right)^5+(-16f_3-48f_1f_2+64f_1f_3+96f_2f_3)\left(\tfrac{r}{a}\right)^6+(108f_2f_3)\left(\tfrac{r}{a}\right)^7+(-80f_1f_3-45f_2^2+160f_3^2)\left(\tfrac{r}{a}\right)^8$$

$$+(-144f_2f_3)\left(\tfrac{r}{a}\right)^{10}+-112f_3^2\left(\tfrac{r}{a}\right)^{12}$$

$$+\cos 2\theta\left\{1.5f_2\left(\tfrac{r}{a}\right)+(4f_3-2f_1)\left(\tfrac{r}{a}\right)^2-6f_2\left(\tfrac{r}{a}\right)^4-12f_3\left(\tfrac{r}{a}\right)^6\right\}$$

$$\frac{\Phi}{\mu^2} = E\left(\frac{a}{r}\right)^4 \Bigg[\frac{1}{4} p_0 \left(\frac{r}{a}\right)^2 + \frac{f_1}{32}(1-2f_1)\left(\frac{r}{a}\right)^4 + 0.02\, f_2 (1-4f_1)\left(\frac{r}{a}\right)^5 + \frac{1}{144}\left(2f_3 - f_1 + 7f_1^2 - 8f_1 f_3 - 4.5 f_2^2\right)\left(\frac{r}{a}\right)^6$$

$$+ \frac{6}{245} f_2 (f_1 - 2f_3)\left(\frac{r}{a}\right)^7 + \frac{1}{768}\left(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3f_2 - 4f_1^2\right)\left(\frac{r}{a}\right)^8 + \frac{1}{63} f_2^2 \left(\frac{r}{a}\right)^9$$

$$+ \frac{1}{400}\left(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2\right)\left(\frac{r}{a}\right)^{10} + \frac{12}{1089} f_2 f_3 \left(\frac{r}{a}\right)^{11} + \frac{1}{3880}\left(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2\right)\left(\frac{r}{a}\right)^{12}$$

$$- \frac{1}{196} f_2 f_3 \left(\frac{r}{a}\right)^{14} - \frac{1}{448} f_3^2 \left(\frac{r}{a}\right)^{16} + \cos 2\theta \Big\{ \frac{1}{20} f_2 \left(\frac{r}{a}\right)^5 + \frac{1}{192}(2f_3 - f_1)\left(\frac{r}{a}\right)^6 - \frac{1}{320} f_2 \left(\frac{r}{a}\right)^8$$

$$- \frac{1}{480} f_3 \left(\frac{r}{a}\right)^{10} + p_2 \left(\frac{r}{a}\right)^2 + n_2 \left(\frac{r}{a}\right)^4 \Big\} \Bigg]$$

$$\frac{1}{r}\frac{\partial \Phi}{\partial r} = E\left(\frac{a}{r}\right)^2 \Bigg[\frac{1}{2} p_0 + \frac{f_1}{8}(1-2f_1)\left(\frac{r}{a}\right)^2 + \frac{1}{10} f_2 (1-4f_1)\left(\frac{r}{a}\right)^3 + \frac{1}{24}\left(2f_3 - f_1 + 4f_1^2 - 8 f_1 f_3 - 4.5 f_2^2\right)\left(\frac{r}{a}\right)^4$$

$$+ \frac{6}{35} f_2 (f_1 - 2f_3)\left(\frac{r}{a}\right)^5 + \frac{1}{96}\left(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3f_2 - 4f_1^2\right)\left(\frac{r}{a}\right)^6 + \frac{1}{7} f_2^2 \left(\frac{r}{a}\right)^7$$

$$+ \frac{1}{40}\left(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2\right)\left(\frac{r}{a}\right)^8 + \frac{12}{99} f_2 f_3 \left(\frac{r}{a}\right)^9 + \frac{1}{240}\left(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$- \frac{1}{14} f_2 f_3 \left(\frac{r}{a}\right)^{12} - \frac{1}{28} f_3^2 \left(\frac{r}{a}\right)^{14} + \cos 2\theta \Big\{ \frac{1}{14} f_2 \left(\frac{r}{a}\right)^3 + \frac{1}{32}(2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{1}{40} f_2 \left(\frac{r}{a}\right)^6$$

$$- \frac{1}{48} f_3 \left(\frac{r}{a}\right)^8 + 2 p_2 + 4 n_2 \left(\frac{r}{a}\right)^2 \Big\} \Bigg]$$

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$$\hat{\sigma}_{rr} = E\left(\frac{a}{R}\right)^2 \Big[\frac{1}{2} q_0 + \frac{f_1}{8}(1-2f_1)\left(\frac{r}{a}\right)^2 + \frac{1}{10} f_2 (1-4f_1)\left(\frac{r}{a}\right)^3 + \frac{1}{24}(2f_3 - f_1 + 4f_1^2 - 8f_1f_3 - 4.5 f_2^2)\left(\frac{r}{a}\right)^4$$

$$+ \frac{6}{35} f_2 (f_1 - 2f_3)\left(\frac{r}{a}\right)^5 + \frac{1}{96}(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3 f_2 - 4 f_1^2)\left(\frac{r}{a}\right)^6 + \frac{1}{7} f_2^2 \left(\frac{r}{a}\right)^7$$

$$+ \frac{1}{40}(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2)\left(\frac{r}{a}\right)^8 + \frac{12}{99} f_2 f_3 \left(\frac{r}{a}\right)^9 + \frac{1}{240}(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2)\left(\frac{r}{a}\right)^{10}$$

$$- \frac{1}{14} f_2 f_3 \left(\frac{r}{a}\right)^{12} - \frac{1}{28} f_3^2 \left(\frac{r}{a}\right)^{14} + \cos 2\theta \Big\{ \frac{1}{70} f_2 \left(\frac{r}{a}\right)^3 + \frac{1}{96}(2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{1}{80} f_2 \left(\frac{r}{a}\right)^6 - \frac{1}{80} f_3 \left(\frac{r}{a}\right)^8$$

$$- 2 p_2 \Big\} \Big]$$

$$\hat{\sigma}_{\theta\theta} = E\left(\frac{a}{R}\right)^2 \Big[\frac{1}{2} q_0 + \frac{3}{8} f_1 (1-2f_1)\left(\frac{r}{a}\right)^2 + \frac{4}{10} f_2 (1-4f_1)\left(\frac{r}{a}\right)^3 + \frac{5}{24}(2f_3 - f_1 + 4f_1^2 - 8f_1 f_3 - 4.5 f_2^2)\left(\frac{r}{a}\right)^4$$

$$+ \frac{36}{35} f_2 (f_1 - 2f_3)\left(\frac{r}{a}\right)^5 + \frac{7}{96}(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3 f_2 - 4 f_1^2)\left(\frac{r}{a}\right)^6 + \frac{8}{7} f_2^2 \left(\frac{r}{a}\right)^7$$

$$+ \frac{9}{40}(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2)\left(\frac{r}{a}\right)^8 + \frac{120}{99} f_2 f_3 \left(\frac{r}{a}\right)^9 + \frac{11}{240}(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2)\left(\frac{r}{a}\right)^{10}$$

$$- \frac{13}{14} f_2 f_3 \left(\frac{r}{a}\right)^{12} - \frac{15}{28} f_3^2 \left(\frac{r}{a}\right)^{14} + \cos 2\theta \Big\{ \frac{20}{70} f_2 \left(\frac{r}{a}\right)^3 + \frac{15}{96}(2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{14}{80} f_2 \left(\frac{r}{a}\right)^6$$

$$- \frac{15}{80} f_3 \left(\frac{r}{a}\right)^8 + 2 p_2 + 12 n_2 \left(\frac{r}{a}\right)^2 \Big\} \Big]$$

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$$\widehat{\Delta\theta} = E\left(\frac{a}{b}\right)^2\left[\sin 2\theta\left\{\frac{1}{70}f_2\left(\frac{a}{a}\right)^3 + \frac{5}{96}(2f_3 - f_1)\left(\frac{a}{a}\right)^4 - \frac{35}{80}f_2\left(\frac{a}{a}\right)^6 - \frac{3}{15}f_3\left(\frac{a}{a}\right)^4 + 2p_2 + 6f_2\left(\frac{a}{a}\right)^2\right\}\right]$$

$$\widehat{\Delta r} - \nu\widehat{\Delta\theta} = E\left(\frac{a}{R}\right)^2\left[\frac{1}{2}(1-\nu)q_{0} + \frac{1-3\nu}{8}f_1(1-2\ldots)\frac{a}{a} + \frac{1-4\nu}{10}f_2(1-4f_1)\left(\frac{a}{a}\right)^3 + \frac{1-5\nu}{24}(2f_3 - f_1 + 4f_2^2 - 8f_1f_3 - 4.5f_3^2)\left(\frac{a}{a}\right)^4\right.$$

$$+ \frac{6(1-6\nu)}{35}f_2(f_1 - 2f_3)\left(\frac{a}{a}\right)^5 + \frac{(1-7\nu)}{96}(12f_1f_2 + 16f_1f_3 - 10f_3^2 - 3f_2 - 4f_1^2)\left(\frac{a}{a}\right)^6 + \frac{(1-8\nu)}{7}f_2^2\left(\frac{a}{a}\right)^7$$

$$+ \frac{(1-9\nu)}{40}(4f_1f_3 + 6f_1f_3 - f_3 - 3\ldots)\left(\frac{a}{a}\right)^8 + \frac{12(1-10\nu)}{99}f_2f_3\left(\frac{a}{a}\right)^9 + \frac{(1-11\nu)}{240}(32f_3^2 - 16f_1f_3 - 9f_2^2)\left(\frac{a}{a}\right)^{10}$$

$$- \frac{(1-13\nu)}{14}f_2f_3\left(\frac{a}{a}\right)^{12} - \frac{(1-15\nu)}{28}f_3^2\left(\frac{a}{a}\right) + \cos 2\theta\left\{\frac{(1-3\nu)}{70}f_2\left(\frac{a}{a}\right)^3 + \frac{(1-15\nu)}{96}(2f_3 - f_1)\left(\frac{a}{a}\right)^4 - \frac{(1-14\nu)}{80}f_2\left(\frac{a}{a}\right)^6\right.$$

$$\left.\left.- \frac{(1-15\nu)}{80}f_3\left(\frac{a}{a}\right)^8 - 2(1+\nu)p_2 - 12(\ldots)\right\}\right]$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2-\frac{1}{2}\left(\frac{\partial w_0}{\partial r}\right)^2=\frac{1}{2}\left(\frac{a}{R}\right)^2\Bigg[(\cos 2\theta-1)\Big\{2f_1\left[1-\left(\tfrac{r}{a}\right)^2\right]+3f_2\left[1-\left(\tfrac{r}{a}\right)^3\right]\left(\tfrac{r}{a}\right)+4f_3\left[1-\left(\tfrac{r}{a}\right)^4\right]\left(\tfrac{r}{a}\right)^2\Big\}$$

$$+\Big\{4f_1^2\left[1-2\left(\tfrac{r}{a}\right)^2+\left(\tfrac{r}{a}\right)^4\right]+12f_1f_2\left[1-\left(\tfrac{r}{a}\right)^2-\left(\tfrac{r}{a}\right)^3+\left(\tfrac{r}{a}\right)^5\right]\left(\tfrac{r}{a}\right)+16f_1f_3\left[1-\left(\tfrac{r}{a}\right)^2-\left(\tfrac{r}{a}\right)^4+\left(\tfrac{r}{a}\right)^6\right]\left(\tfrac{r}{a}\right)^2$$

$$+9f_2^2\left[1-2\left(\tfrac{r}{a}\right)^3+\left(\tfrac{r}{a}\right)^6\right]\left(\tfrac{r}{a}\right)^2+24f_2f_3\left[1-\left(\tfrac{r}{a}\right)^3-\left(\tfrac{r}{a}\right)^4+\left(\tfrac{r}{a}\right)^7\right]\left(\tfrac{r}{a}\right)^3+16f_3^2\left[1-2\left(\tfrac{r}{a}\right)^4+\left(\tfrac{r}{a}\right)^8\right]\left(\tfrac{r}{a}\right)^4\Big\}\Bigg]$$

$$=\frac{1}{2}\left(\frac{a}{R}\right)^2\Bigg[\cos 2\theta\Big\{2f_1\left(\tfrac{r}{a}\right)^2+3f_2\left(\tfrac{r}{a}\right)^3+2(2f_3-f_1)\left(\tfrac{r}{a}\right)^4-3f_2\left(\tfrac{r}{a}\right)^6-4f_3\left(\tfrac{r}{a}\right)^8\Big\}$$

$$+2f_1(2f_1-1)\left(\tfrac{r}{a}\right)^2+(12f_1f_2-3f_2)\left(\tfrac{r}{a}\right)^3+(-8f_1^2+16f_1f_3+9f_2^2-4f_3+2f_1)\left(\tfrac{r}{a}\right)^4$$

$$+(-12f_1f_2+24f_2f_3)\left(\tfrac{r}{a}\right)^5+(4f_1^2-12f_1f_2-16f_1f_3+10f_3^2+3f_3)\left(\tfrac{r}{a}\right)^6$$

$$-18f_2^2\left(\tfrac{r}{a}\right)^7+\left(12f_1f_2-16f_1f_3\overset{-4f_3}{-2}+24f_2f_3\right)\left(\tfrac{r}{a}\right)^8+(-24f_2f_3)\left(\tfrac{r}{a}\right)^9+(16f_1f_3+9f_2^2-32f_3^2)\left(\tfrac{r}{a}\right)^{10}$$

$$+24f_2f_3\left(\tfrac{r}{a}\right)^{12}+16f_3^2\left(\tfrac{r}{a}\right)^{14}\Big\}\Bigg]$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2-\frac{1}{2}\left(\frac{\partial w_0}{\partial r}\right)^2=\left(\frac{a}{R}\right)^2\Big[\cos 2\theta\Big\{f_1\left(\frac{r}{a}\right)+\frac{3}{2}f_2\left(\frac{r}{a}\right)^3+(2f_3-f_1)\left(\frac{r}{a}\right)^4-\frac{3}{2}f_2\left(\frac{r}{a}\right)^6-2f_3\left(\frac{r}{a}\right)^8\Big\}$$

$$-f_1(1-2f_1)\left(\frac{r}{a}\right)^2-\frac{3}{2}f_2(1-4f_2)\left(\frac{r}{a}\right)^3-\left(2f_3-f_1+4f_1^2-8f_1f_3-4.5f_2^2\right)\left(\frac{r}{a}\right)^4$$

$$-6f_2(f_1-2f_3)\left(\frac{r}{a}\right)^5+\frac{1}{2}\left(12f_1f_2+16f_1f_3-16f_3^2-3f_2-4f_1^2\right)\left(\frac{r}{a}\right)^6-9f_2^2\left(\frac{r}{a}\right)^7$$

$$-2\left(4f_1f_3+6f_2f_3-f_3-3f_1f_2\right)\left(\frac{r}{a}\right)^8-12f_2f_3\left(\frac{r}{a}\right)^9-\frac{1}{2}\left(32f_3^2-16f_1f_3-9f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$+12f_2f_3\left(\frac{r}{a}\right)^{12}+8f_3^2\left(\frac{r}{a}\right)^{14}\Big\}\Big]$$

$$\frac{\partial u}{\partial r}=\left(\frac{a}{R}\right)^2\Big[\frac{1}{2}(1-\nu)q_0+\frac{3(3-\nu)}{8}f_1(1-2f_1)\left(\frac{r}{a}\right)^2+\frac{4(4-\nu)}{10}f_2(1-4f_1)\left(\frac{r}{a}\right)^3$$

$$+\frac{5(5-\nu)}{24}\left(2f_3-f_1+4f_1^2-8f_1f_3-4.5f_2^2\right)\left(\frac{r}{a}\right)^4+\frac{36(6-\nu)}{35}f_2(f_1-2f_3)\left(\frac{r}{a}\right)^5$$

$$+\frac{7(7-\nu)}{96}\left(12f_1f_2+16f_1f_3-16f_3^2-3f_2-4f_1^2\right)\left(\frac{r}{a}\right)^6+\frac{8(8-\nu)}{7}f_2^2\left(\frac{r}{a}\right)^7$$

$$+\frac{9(9-\nu)}{40}\left(4f_1f_3+6f_2f_3-f_3-3f_1f_2\right)\left(\frac{r}{a}\right)^8+\frac{120(10-\nu)}{99}f_2f_3\left(\frac{r}{a}\right)^9+\frac{11(11-\nu)}{240}\left(32f_3^2-16f_1f_3-9f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$-\frac{13(13-\nu)}{14}f_2f_3\left(\frac{r}{a}\right)^{12}-\frac{15(15-\nu)}{28}f_3^2\left(\frac{r}{a}\right)^{14}$$

$$- \cos 2\theta \Big\{ f_1 \left(\frac{a}{a}\right)^2 + \frac{(104+20\nu)}{70} f_2 \left(\frac{a}{a}\right)^4 + \frac{(95+15\nu)}{96} (2f_3 - f_1)\left(\frac{a}{a}\right)^5 - \frac{(119+14\nu)}{80} f_2 \left(\frac{a}{a}\right)^6 - \frac{(159+15\nu)}{80} f_3 \left(\frac{a}{a}\right)^8$$

$$+ 2(1+\nu) f_1 c_2 + 12\nu A_2 \left(\frac{a}{a}\right)^2 \Big\} \Big]$$

$$\frac{u}{r} = \left(\frac{a}{r}\right)^3 \Big[\frac{1}{2}(1-\nu)\frac{a}{(a)} + \frac{(3-\nu)}{8} f_1 (1-2f_1)\left(\frac{a}{a}\right)^2 + \frac{(4-\nu)}{10} f_2 (1-2f_1)\left(\frac{a}{a}\right)^3 + \frac{(5-\nu)}{24}(2f_3 - f_1 + 4f_1^2 - 8f_1 f_3 - 4.5 f_2^2)\left(\frac{a}{a}\right)^5$$

$$+ \frac{6(6-\nu)}{35} f_2 (f_1 - 2f_3)\left(\frac{a}{a}\right)^6 + \frac{(7-\nu)}{16}(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3f_2 - 4f_1^2)\left(\frac{a}{a}\right)^7 + \frac{(8-\nu)}{7} f_2^2 \left(\frac{a}{a}\right)^8$$

$$+ \frac{(9-\nu)}{40}(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2)\left(\frac{a}{a}\right)^9 + \frac{15(10-\nu)}{91} f_2 f_3 \left(\frac{a}{a}\right)^{10} + \frac{(11-\nu)}{240}(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2)\left(\frac{a}{a}\right)^{11}$$

$$- \frac{(13-\nu)}{14} f_2 f_3 \left(\frac{a}{a}\right)^{13} - \frac{(15-\nu)}{28} f_3^2 \left(\frac{a}{a}\right)^{15}$$

$$- \cos 2\theta \Big\{ \frac{1}{3} f_1 \left(\frac{a}{a}\right)^3 + \frac{(26+5\nu)}{70} f_2 \left(\frac{a}{a}\right)^4 + \frac{(17+3\nu)}{96}(2f_3 - f_1)\left(\frac{a}{a}\right)^5 - \frac{(17+2\nu)}{80} f_2 \left(\frac{a}{a}\right)^7 - \frac{(53+5\nu)}{240} f_3 \left(\frac{a}{a}\right)^9$$

$$+ 2(1+\nu) f_2 \left(\frac{a}{a}\right) + 4\nu A_2 \left(\frac{a}{a}\right)^3 \Big\} \Big]$$

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$$\frac{1}{a}\frac{\partial v}{\partial \theta} = \left(\frac{a}{R}\right)^2 \Big[\cos 2\theta \Big\{ \frac{1}{3} f_1 \left(\frac{r}{a}\right)^2 + \frac{(23+2\nu)}{35} f_2 \left(\frac{r}{a}\right)^3 + \frac{(17+\nu)}{48} (2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{(31+\nu)}{80} f_2 \left(\frac{r}{a}\right)^6$$

$$- \frac{(49+\nu)}{120} f_3 \left(\frac{r}{a}\right)^8 + 4(1+\nu) p_2 + 4(3+\nu) r_2 \left(\frac{r}{a}\right)^2 \Big\} \Big]$$

$$\frac{v}{R} = \left(\frac{a}{R}\right)^3 \Big[\sin 2\theta \Big\{ \frac{1}{6} f_1 \left(\frac{r}{a}\right)^3 + \frac{(23+2\nu)}{70} f_2 \left(\frac{r}{a}\right)^4 + \frac{(17+\nu)}{96} (2f_3 - f_1)\left(\frac{r}{a}\right)^5 - \frac{(31+\nu)}{160} f_2 \left(\frac{r}{a}\right)^7$$

$$- \frac{(49+\nu)}{240} f_3 \left(\frac{r}{a}\right)^8 + 2(1+\nu) p_2 \left(\frac{r}{a}\right) + 2(3+\nu) r_2 \left(\frac{r}{a}\right)^3 \Big\} \Big]$$

Thus the non-uniform portions are (i.e. terms independent of [illegible], at $r = a$)

$$\hat{n}_a = \cos 2\theta \left\{ \frac{f_2}{560} - \frac{f_1}{96} + \frac{f_3}{120} - 2 p_2 \right\}$$

$$\hat{t}_a = \cos 2\theta \left\{ \frac{31}{280} f_2 - \frac{5}{32} f_1 + \frac{1}{8} f_3 + 2 p_2 + 12 r_2 \right\}$$

$$\hat{r}_a = \sin 2\theta \left\{ \frac{395}{560} f_2 - \frac{5}{96} f_1 + \frac{1}{15} f_3 + 2 p_2 + 6 r_2 \right\}$$

$$\frac{u}{r}\Big|_{\alpha} = -\cos 2\theta \left\{ \frac{(13-3\nu)}{96} f_1 + \frac{(89+36\nu)}{560} f_2 + \frac{(21+5\nu)}{120} f_3 + 2(1+\nu) p_2 + 4\nu \lambda_2 \right\}$$

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$$\frac{v}{r}\Big|_{\alpha} = \sin 2\theta \left\{ -\frac{(1+\nu)}{96} f_1 + \frac{(15+115\nu)}{1120} f_2 + \frac{(9+\nu)}{60} f_3 + 2(1+\nu) p_2 + 2(3+\nu) \lambda_2 \right\}$$

$$\frac{t}{2} - 6 q_2 - 4 s_2 = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2 \left\{ \frac{f_2}{560} - [illegible] + \frac{f_3}{120} - 2 p_2 \right\}$$

$$6 q_2 - \frac{t}{2} = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2 \left\{ \frac{31}{280} f_2 - \frac{5}{32} f_1 + \frac{1}{8} f_3 + 4 p_2 + 12 \lambda_2 \right\}$$

$$\frac{t}{2} + 6 q_2 + 2 s_2 = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2 \left\{ -\frac{79}{1120} f_2 + \frac{5}{16} f_1 - \frac{1}{15} f_3 - 2 p_2 - 6 \lambda_2 \right\}$$

$$\frac{t}{2}(1+\nu) + 2(1+\nu) q_2 + 4 s_2 = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2 \left\{ - \frac{[illegible]}{[illegible]} f_1 - \frac{(89+36\nu)}{560} f_2 - \frac{(21+5\nu)}{120} f_3 - 2(1+\nu) p_2 - 4\nu \lambda_2 \right\}$$

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$$2(1+\nu) q_2 - \frac{t}{2}(1+\nu) = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2 \left\{ -\frac{(1+\nu)}{96} f_1 + \frac{(15+115\nu)}{1120} f_2 + \frac{(19+\nu)}{60} f_3 + 2(1+\nu) p_2 + 2(3+\nu) \lambda_2 \right\}$$

[illegible]

$1)\ q_2 + 0.666667 S_2 - 0.083333 = \eta \{ 0.333333 \tfrac{1}{p_2} + 0 + 0.00173611 f_1 - 0.00029762 f_2 - 0.00138889 f_3 \}$

$2)\ q_2 + 0 - 0.083333 = \eta \{ 0.333333 \tfrac{1}{p_2} + 2\lambda_1 - 0.02604167 f_1 + 0.01845238 f_2 + 0.02083333 f_3 \}$

$3)\ q_2 + 0.333333 S_2 + 0.083333 = \eta \{ -0.333333 \tfrac{1}{p_2} - \lambda_3 + 0.00868056 f_1 - 0.01175595 f_2 - 0.01111111 f_3 \}$

$4)\ q_2 + 1.5384615 S_2 + 0.25000 = \eta \{ - p_2 - 0.461538 \lambda_1 - 0.04847756 f_1 - 0.06648352 f_2 - 0.07211538 f_3 \}$

$5)\ q_2 + 0 - 0.350000 = \eta \{ p_2 + 2.5384628 \lambda_2 - 0.05530633 f_1 + 0.03442995 f_2 + 0.05961538 f_3 \}$

$0.666667 S_2 + 0 = \eta \{ 0 - 3\lambda_2 + 0.03717778 f_1 - 0.01875000 f_2 - 0.02222222 f_3 \}$

$0.333333 S_2 + 0.166667 = \eta \{ -0.666667 p_2 - 3\lambda_3 + 0.03472222 f_1 - 0.03020833 f_2 - 0.03194444 f_3 \}$

$1.2051282 S_2 + 0.166667 = \eta \{ -0.666667 p_2 + 1.5384462 \lambda_2 - 0.0571812 f_1 - 0.05472757 f_2 - 0.06100427 f_3 \}$

$1.5384615 S_2 + 0.5000 = \eta \{ -2 p_2 - 3\lambda_2 - 0.04326923 f_1 - 0.12091347 f_2 - 0.13173076 f_3 \}$

$6)\ S_2 + 0 = \eta \{ 0 - 3\lambda_2 + 0.04166666 f_1 - 0.02812500 f_2 - 0.03333333 f_3 \}$

$7)\ S_2 + 0.50000 = \eta \{ -2 p_2 - 9\lambda_2 + 0.10416667 f_1 - 0.09062500 f_2 - 0.09583333 f_3 \}$

$8)\ S_2 + 0.1382978 = \eta \{ -0.5531915 p_2 + 0.4468071 \lambda_1 - 0.04742907 f_1 - 0.04541224 f_2 - 0.05069056 f_3 \}$

$9)\ S_2 + 0.32500 = \eta \{ -1.3 p_2 - 1.75\lambda_2 - 0.02812500 f_1 - 0.07859376 f_2 - 0.08562500 f_3 \}$

$$0.500000 = \eta\{-2p_2 - 6r_2 + 0.06250000 f_1 - 0.06250000 f_2 - 0.06250000 f_3\}$$

$$0.3617022 = \eta\{-1.4468087 p_2 - 9.446809 r_2 + 0.1515 9545 f_1 - 0.04521276 f_2 - 0.04521277 f_3\}$$

$$0.1867022 = \eta\{-0.7468085 p_2 - 2.396809 r_2 + 0.01930408 f_1 - 0.03318152 f_2 - 0.03500444 f_3\}$$

$$0.350000 = \eta\{-p_2 - 3r_2 + 0.03125000 f_1 - 0.03125000 f_2 - 0.03125000 f_3\}$$

$$0.250000 = \eta\{-p_2 - 6.529412 r_2 + 0.10477942 f_1 - 0.03125000 f_2 - 0.03125000 f_3\}$$

$$0.250000 = \eta\{-p_2 - 3.209103 r_2 + 0.02584877 f_1 - 0.04443110 f_2 - 0.04687205 f_3\}$$

$$3.529412\, r_2 = 0.07352942 f_1 - 0 f_2 + 0 f_3$$

$$3.320009\, r_2 = 0.07893065 f_1 + 0.01318110 f_2 + 0.01562205 f_3$$

$$\boxed{r_2 = 0.02083333 f_1 - 0} + 0 f_3$$

$$r_2 = 0.02377423 f_1 + 0.00397020 f_2 + 0.00470542 f_3$$

$$0 = 0.00294090 f_1 + 0.00397020 f_2 + 0.00470542 f_3$$

$$\boxed{-f_3 = 0.625000 f_1 + 0.843750 f_2}$$

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$$2\eta p_2 = -0.500000 + \eta \left\{ \begin{array}{ll} f_1 & f_2 \\ -0.1985294 & \\ +0.1360294 & -0.0625000 \\ +0.0390625 & +0.0527344 \end{array} \right\}$$

$$\boxed{\eta p_2 = -0.250000 - \eta \{ 0.0117188\, f_1 + 0.0048980\, f_2 \}}$$

$$3s_2 = \eta \left\{ \begin{array}{ll} f_1 & f_2 \\ +0.0299303 & +0.0124667 \\ -0.2406915 & \\ +0.0984043 & -0.1641622 \\ +0.1123670 & +0.1516955 \end{array} \right\}$$

$$\boxed{s_2 = -\eta \{ 0 + 0 \}}$$

$$4q_2 = \eta \left\{ \begin{array}{ll} f_1 & f_2 \\ +0.0078125 & +0.0032552 \\ +0.0112179 & \\ -0.0641026 & -0.0[illegible] \\ +0.0398638 & +0.0538161 \end{array} \right\}$$

$$\boxed{q_2 = -\eta \{ 0.0013021\, f_1 + 0.0007534\, f_2 \}}$$

CHECK

f_1	f_2
−0.0117188	−0.0048980
+0.0521846	− 0
−0.0052083	+0.0542995
−0.0372596	−0.0503048

O.K.

(s_2 = 0.0208333 ...)

$$\frac{t}{2} + \nu_0 = \eta\Big\{\frac{1}{2} p_0 + \frac{3}{8} f_1(1-2f_1) + \frac{3}{5} f_2(1-4f_1) + \frac{1}{24}(2f_3 - f_1 + 4f_1^2 - 8f_1 f_3 - 4.5 f_2^2)$$

$$+ \frac{6}{35} f_2(f_1 - 2f_3) + \frac{1}{96}(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3f_2 - 4f_1^2) + \frac{1}{7} f_2^2$$

$$+ \frac{1}{40}(4f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2) + \frac{12}{99} f_2 f_3 + \frac{1}{240}(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2) - \frac{1}{4} f_2 f_3 - \frac{1}{24} f_3^2$$

$$\frac{t}{2} - \nu_0 = \eta\Big\{\frac{1}{2} p_0 + \frac{3}{8} f_1(1-2f_1) + \qquad \Big\}$$

$$p_0 = \frac{t}{\eta} - \frac{1}{2} f_1(1-2f_1) - \frac{1}{2} f_2(1-4f_1) - \frac{1}{4}(2f_3 - f_1 + 4f_1^2 - 8 f_1 f_3 - 4.5 f_2^2)$$

$$- \frac{6}{5} f_2(f_1 - 2f_3) - \frac{1}{12}(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3f_2 - 4f_1^2) - \frac{9}{7} f_2^2$$

$$- \frac{1}{4}(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2) - \frac{4}{3} f_2 f_3 - \frac{1}{20}(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2) + f_2 f_3 + \frac{4}{7} f_3^2$$

$$\nu_0 = -\eta\Big\{\frac{1}{8} f_1(1-2f_1) + \frac{3}{20} f_2(1-4f_1) + \frac{1}{12}(2f_3 - f_1 + 4f_1^2 - 8 f_1 f_3 - 4.5 f_2^2)$$

$$+ \frac{3}{7} f_2(f_1 - 2f_3) + \frac{1}{32}(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3f_2 - 4f_1^2) + \frac{1}{2} f_2^2 + \frac{1}{10}(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2)$$

$$+ \frac{6}{11} f_2 f_3 + \frac{1}{48}(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2) - \frac{3}{7} f_2 f_3 - \frac{1}{4} f_3^2\Big\}$$

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where $f_1(1-2f_1) = A$, $\quad f_2(1-4f_2) = B$, $\quad (12f_3 - f_1 + 4f_1^2 - 8f_1f_3 - 45f_2^2) = C$,

$f_2(f_1 - 2f_3) = D$, $\quad (12f_1 + 16f_3^2 - 10f_1^2 - 3f_2 - 4f_1^2) = E$, $\quad f_2^2 = F$,

$(4f_1f_3 + 6f_2f_3 - f_3 - 3f_1f_2) = G$, $\quad f_2f_3 = H$, $\quad 32f_3^2 - 16f_1f_3 - 9f_2^2) = I$,

$f_2f_3 = J$, $\quad f_3^2 = K$

$$\widehat{rr} + \widehat{\theta\theta} = E\left(\frac{a}{R}\right)^2 \Big[\Big\{ \frac{1}{7} - \frac{1}{2}A - \frac{1}{2}B - \frac{1}{4}C - \frac{6}{5}D - \frac{1}{12}E - \frac{9}{7}F - \frac{1}{4}G - \frac{4}{3}H - \frac{1}{20}I$$

$$+ J + \frac{4}{7}K \Big\} + \frac{1}{2}A\left(\frac{r}{a}\right)^2 + \frac{1}{2}B\left(\frac{r}{a}\right)^2 + \frac{1}{4}C\left(\frac{r}{a}\right)^4 + \frac{6}{5}D\left(\frac{r}{a}\right)^5 + \frac{1}{12}E\left(\frac{r}{a}\right)^6 + \frac{9}{7}F\left(\frac{r}{a}\right)^7$$

$$+ \frac{1}{4}G\left(\frac{r}{a}\right)^8 + \frac{4}{3}H\left(\frac{r}{a}\right)^9 + \frac{1}{20}I\left(\frac{r}{a}\right)^{10} - J\left(\frac{r}{a}\right)^{12} - \frac{4}{7}K\left(\frac{r}{a}\right)^{14} +$$

$$\cos 2\theta \Big\{ \frac{3}{10} f_2\left(\frac{r}{a}\right)^3 + \frac{1}{6}(2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{3}{16} f_2\left(\frac{r}{a}\right)^6 - \frac{1}{5} f_3\left(\frac{r}{a}\right)^8 + 0 \quad + 12 n_2\left(\frac{r}{a}\right)^2 \Big\} \Big]$$

$$\int_0^{2\pi} (\widehat{rr} + \widehat{\theta\theta})^2 d\theta = \pi E^2 \left(\frac{a}{R}\right)^4 \Big[2\Big\{ g_0^2 + A g_0\left(\frac{r}{a}\right)^2 + B g_0\left(\frac{r}{a}\right)^3 + \left(\frac{A^2}{4} + \frac{1}{2}C g_0\right)\left(\frac{r}{a}\right)^4 + \left(\frac{1}{2}AB + \frac{12}{5}D g_0\right)\left(\frac{r}{a}\right)^5$$

$$+ \left(\frac{1}{4}B^2 + \frac{1}{6}E g_0\right)\left(\frac{r}{a}\right)^6 + \left(\frac{1}{4}BC + \frac{18}{7}F g_0\right)\left(\frac{r}{a}\right)^7 + \left(\frac{1}{16}C^2 + \frac{1}{2}G g_0\right)\left(\frac{r}{a}\right)^8 + ($$

$$(A\ldots(CD))^2 A\ldots = \pi E^2 \left(\frac{a}{R}\right)^4 2\Big\{ q_0^3 + A q_0^2 \left(\frac{a}{a}\right)^2 + B q_0 \left(\frac{a}{a}\right)^3 + \left(\frac{A^2}{4} + \frac{1}{2} C q_0\right)\left(\frac{a}{a}\right)^4 + \left(\frac{1}{2} AB + \frac{12}{5} D q_0\right)\left(\frac{a}{a}\right)^5$$

$$+\left(\frac{1}{4} B^2 + \frac{1}{4} AC + \frac{1}{6} E q_0\right)\left(\frac{a}{a}\right)^6 + \left(\frac{1}{4} BC + \frac{6}{5} AD + \frac{16}{7} F q_0\right)\left(\frac{a}{a}\right)^7 + \left(\frac{1}{16} C^2 + \frac{6}{5} BD + \frac{1}{12} AE + \frac{1}{2} G q_0\right)\left(\frac{a}{a}\right)^8$$

$$+\left(\frac{3}{5} CD + \frac{1}{12} BE + \frac{9}{7} AF + \frac{1}{3} H q_0\right)\left(\frac{a}{a}\right)^9 + \left(\frac{36}{25} D^2 + \frac{1}{24} CE + \frac{9}{7} BF + \frac{1}{4} AG + \frac{1}{10} I q_0\right)\left(\frac{a}{a}\right)^{10}$$

$$+\left(\frac{1}{5} DE + \frac{9}{7} CF + \frac{1}{4} BG + \frac{4}{3} AH\right)\left(\frac{a}{a}\right)^{11} +$$

$$+\left(\frac{1}{144} E^2 + \frac{108}{35} DF + \frac{1}{8} CG + \frac{2}{3} BH + \frac{1}{10} AI - [illegible] q_0\right)\left(\frac{a}{a}\right)^{12} + \left(\frac{3}{14} EF + \frac{3}{5} DG + \frac{2}{3} CH$$

$$+ \frac{1}{20} BI\right)\left(\frac{a}{a}\right)^{13} + \left(\frac{81}{49} F^2 + \frac{1}{16} EG + \frac{44}{15} DH + \frac{1}{40} CI - [illegible] - \frac{8}{7} K [illegible]\right)\left(\frac{a}{a}\right)^{14}$$

$$+\left(\frac{9}{14} FG + \frac{4}{18} EH + \frac{3}{50} DI - [illegible]\right)\left(\frac{a}{a}\right)^{15} +$$

$$+\left(\frac{1}{16} G^2 + \frac{24}{7} FH + \frac{1}{120} EI - \frac{1}{2} CJ - \frac{4}{7} AK\right)\left(\frac{a}{a}\right)^{16} + \left(\frac{2}{3} GH + \frac{9}{70} FI - \frac{12}{5} DJ - \frac{4}{7} BK\right)\left(\frac{a}{a}\right)^{17}$$

$$+\left(\frac{16}{9} H^2 + \frac{1}{40} GI - \frac{1}{6} EJ - \frac{8}{7} CK\right)\left(\frac{a}{a}\right)^{18} + \left(\frac{1}{30} HI - \frac{16}{7} FJ - \frac{48}{35} DK\right)\left(\frac{a}{a}\right)^{19}$$

$$+\left(\frac{1}{400} I^2 - \frac{1}{2} GJ - \frac{2}{21} EK\right)\left(\frac{a}{a}\right)^{20} + \left(-\frac{4}{3} HJ - \frac{72}{49} FK\right)\left(\frac{a}{a}\right)^{21} + \left(-\frac{1}{10} IJ - \frac{2}{7} GK\right)\left(\frac{a}{a}\right)^{22}$$

$$+\left(J^2 - \frac{4}{70} IK\right)\left(\frac{a}{a}\right)^{24} + \frac{8}{7} JK \left(\frac{a}{a}\right)^{26} + \frac{16}{49} K^2 \left(\frac{a}{a}\right)^{28} - \frac{32}{21} HK \left(\frac{a}{a}\right)^{23}$$

$$+ \Big\{ \frac{9}{120} A_2^2 \left(\frac{a}{a}\right)^6 + \frac{1}{10} A_2 (2 A_3 - [illegible]) \left(\frac{a}{a}\right)^7 + \left[\frac{1}{36} [illegible] \right]^2 - \frac{9}{2} A_2 A_2 \Big] \left(\frac{a}{a}\right)^8$$

$$+\Big\{ 1 + 4 f_2^2 \left(\tfrac{\rho}{a}\right)^4 + \tfrac{3}{5} f_2 f_3 \left(\tfrac{\rho}{a}\right)^5 + \left[\tfrac{9}{100} f_3^2 + 4 f_2 (\ldots f_1 f_2)\right]\left(\tfrac{\rho}{a}\right)^6 + \tfrac{1}{10} f_2 (2 f_3 - \ldots$$

$$+ \left[\tfrac{1}{36}(2 f_3 - f_1)^2 - \tfrac{9}{2} f_2 f_2\right]\left(\tfrac{\rho}{a}\right)^8 - \tfrac{9}{80} f_2^2 \left(\tfrac{\rho}{a}\right)^9 - \left[\tfrac{1}{16} f_2 (2 f_3 - f_1) + \tfrac{28}{5} f_2 f_3\right]\left(\tfrac{\rho}{a}\right)^{10}$$

$$- \tfrac{6}{50} f_2 f_3 \left(\tfrac{\rho}{a}\right)^{11} + \left[\tfrac{9}{25} f_2^2 - \tfrac{1}{15} f_3 (2 f_3 - f_1)\right]\left(\tfrac{\rho}{a}\right)^{12} + \tfrac{3}{40} f_2 f_3 \left(\tfrac{\rho}{a}\right)^{14} + \tfrac{1}{25} f_3^2 \left(\tfrac{\rho}{a}\right)^{16} \Big\}$$

$$\int_0^{2\pi} \hat{u} \hat{u}\, d\theta = \pi E^2 \left(\tfrac{a}{R}\right)^4 \Big[2 \Big\{ \tfrac{1}{4} q_0^2 + \tfrac{1}{4} A q_0 \left(\tfrac{\rho}{a}\right)^2 + \tfrac{1}{4} B q_0 \left(\tfrac{\rho}{a}\right)^3 + \left(\tfrac{3}{64} A^2 + \tfrac{1}{8} C q_0\right)\left(\tfrac{\rho}{a}\right)^4$$

$$+ \left(\tfrac{7}{80} AB + \tfrac{3}{5} D q_0\right)\left(\tfrac{\rho}{a}\right)^5 + \left(\tfrac{4}{100} B^2 + \tfrac{1}{24} AC + \tfrac{1}{24} E q_0\right)\left(\tfrac{\rho}{a}\right)^6 + \left(\tfrac{3}{80} BC + \tfrac{27}{140} AD + \tfrac{9}{14} F q_0\right)\left(\tfrac{\rho}{a}\right)^7$$

$$+ \left(\tfrac{5}{24^2} C^2 + \tfrac{1}{35} BD + \tfrac{5}{48 \cdot 8} AE + \tfrac{1}{8} G q_0\right)\left(\tfrac{\rho}{a}\right)^8 + \left(\tfrac{11}{140} CD + \tfrac{11}{960} BE + \tfrac{11}{56} AF + \tfrac{11}{14} H q_0\right)\left(\tfrac{\rho}{a}\right)^9$$

$$+ \left(\tfrac{3}{35^2} D^2 + \tfrac{1}{192} CE + \tfrac{12}{70} BF + \tfrac{1}{80} AG + \tfrac{1}{42} I q_0\right)\left(\tfrac{\rho}{a}\right)^{10} + \left(\tfrac{13}{35 \cdot 16} DE + \tfrac{13}{2447} CF + \tfrac{13}{400} BG + \tfrac{13}{66} AH\right)\left(\tfrac{\rho}{a}\right)^{11}$$

$$+ \left(\tfrac{7}{48^2} E^2 + \tfrac{12}{35} DF + \tfrac{7}{480} CG + \tfrac{28}{140} BH + \tfrac{7}{160} AI - \tfrac{1}{7} J q_0\right)\left(\tfrac{\rho}{a}\right)^{12} +$$

$$+ \left(\tfrac{5}{3247} EF + \tfrac{9}{140} DG + \tfrac{5}{64} CH + \tfrac{1}{\ldots} BI\right)\left(\tfrac{\rho}{a}\right)^{13} + \left(\tfrac{1}{49} F^2 + \tfrac{1}{240} EG + \tfrac{1}{385} DH + \tfrac{1}{160} CI - \tfrac{1}{7} AJ - \tfrac{2}{7} K q_0\right)\left(\tfrac{\rho}{a}\right)^{14}$$

$$+ \left(\tfrac{11}{280} FG + \tfrac{17}{33 \cdot 24} EH + \tfrac{17}{1400} DI - \tfrac{17}{140} BJ\right)\left(\tfrac{\rho}{a}\right)^{15} +$$

$$+\left(\frac{9}{1600}G^2+\frac{24}{77}FH+\frac{1}{1280}EI-\frac{3}{56}CJ-\frac{9}{112}AK\right)\left(\frac{2}{a}\right)^{16}$$

$$+\left(\frac{19}{330}GH+\frac{19}{7\times 240}FI-\frac{57}{245}DJ-\frac{19}{480}BK\right)\left(\frac{2}{a}\right)^{17}$$

$$+\left(\frac{160}{1089}H^2+\frac{1}{480}GI-\frac{5}{24\times 14}EJ-\frac{5}{24\times 7}CK\right)\left(\frac{2}{a}\right)^{18}$$

$$+\left(\frac{1}{660}HI-\frac{3}{14}FJ-\frac{9}{70}DK\right)\left(\frac{2}{a}\right)^{19}+\left(\frac{11}{240^2}I^2-\frac{11}{480}\,\frac{11}{280}GJ-\frac{11}{14\times 16}EK\right)\left(\frac{2}{a}\right)^{20}$$

$$+\left(\frac{46}{231}HJ-\frac{23}{28\times 7}FK\right)\left(\frac{2}{a}\right)^{21}+\left(-\frac{1}{140}IJ-\frac{3}{140}GK\right)\left(\frac{2}{a}\right)^{22}$$

$$+\left(\frac{13}{14^2}J^2-\frac{13}{190\times 28}IK\right)\left(\frac{2}{a}\right)^{24}+\frac{1}{14}JK\left(\frac{2}{a}\right)^{26}+\frac{15}{28^2}K^2\left(\frac{2}{a}\right)^{28}-\frac{25}{231}HK\left(\frac{2}{a}\right)^{23}$$

$$+\Big\{-f_2^2-24f_2a_2\left(\frac{2}{a}\right)^2-\frac{19}{35}f_2b_2\left(\frac{2}{a}\right)^3-\frac{3}{24}(2f_3-f_1)f_2\left(\frac{2}{a}\right)^4+\left[\frac{13}{40}f_3\,[\text{illegible}]+\frac{1}{8}(2f_3-f_1)f_3\right]\left(\frac{2}{a}\right)^4$$

$$+\frac{1}{4800}f_2^2\qquad+\frac{6}{5}f_2\,[\text{illegible}]\left(\frac{2}{a}\right)^5$$

$$+\frac{1}{192}f_2(2f_3-f_1)\left(\frac{2}{a}\right)^7+\left[\frac{1}{[\text{illegible}]}[\text{illegible}]-\frac{3}{20}f_2b_2+\frac{7}{20}f_3b_2\right]\left(\frac{2}{a}\right)^5$$

$$-\frac{7}{2800}f_2^2\left(\frac{2}{a}\right)^{[\text{illegible}]}-\left[\frac{29}{9\times 112}f_2(2f_3-f_1)+\frac{3}{30}[\text{illegible}]\right]\left(\frac{2}{a}\right)^{[\text{illegible}]}-\frac{1}{160}f_1f_3\left(\frac{2}{a}\right)$$

$$+\left[\frac{14}{6400}f_2^2-\frac{1}{8\times 32}f_3(2f_3-f_1)\,[\text{illegible}]\right]^{12}+\frac{[\text{illegible}]}{64\,[\text{illegible}]}[\text{illegible}]+\frac{15}{6400}f_3^2\left(\frac{2}{a}\right)^{16}\Big\}$$

$$\int_0^{\pi} \hat{w}^2 \, d\theta = E^2 \left(\frac{a}{R}\right)^4 \pi \Big[4 p_2^2 + 24 p_2 [illegible] \left(\frac{a}{a}\right)^2 + \frac{16}{35} f_2 p_2 \left(\frac{a}{a}\right)^3 + \left\{ 36 n_2^2 + \frac{5}{24} (2f_3 - f_1) p_2 \right\} \left(\frac{a}{a}\right)^4$$

$$+ \frac{48}{35} f_2 n_2 \left(\frac{a}{a}\right)^5 + \left\{ \frac{64}{4900} f_2^2 + \frac{40}{8} (2f_3 - f_1) n_2 \right\} \left(\frac{a}{a}\right)^6 + \frac{5}{420} f_2 (2f_3 - f_1) \left(\frac{a}{a}\right)^7$$

$$+ \left\{ \frac{25}{16^2} (2f_3 - f_1)^2 - \frac{105}{90} f_2 n_2 - \frac{3}{20} f_3 p_2 \right\} \left(\frac{a}{a}\right)^8 - \frac{1}{100} f_2^2 \left(\frac{a}{a}\right)^9 - \left\{ \frac{7}{96 \times 16} f_2 (2f_3 - f_1)\right.$$

$$\left. + \frac{9}{30} f_3 n_2 \right\} \left(\frac{a}{a}\right)^{10} - \frac{3}{350} f_2 f_3 \left(\frac{a}{a}\right)^{11} + \left\{ \frac{3.5^2}{80^2} f_2^2 - \frac{3}{8 \times 96} f_3 (2f_3 - f_1) \right\} \left(\frac{a}{a}\right)^{12}$$

$$+ \frac{21}{40^2} f_2 f_3 \left(\frac{a}{a}\right)^{14} + \frac{9}{6400} [illegible] \left(\frac{a}{a}\right)^{16} \Big]$$

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the part from non-uniform part

$$\int_0^{\pi} \left[\hat{w} (\hat{w} - \hat{w}^2)\right] d\theta = E^2 \left(\frac{a}{R}\right)^4 \pi \Big[-8 p_2^2 - 18 p_2 n_2 \left(\frac{a}{a}\right)^2 - f_2 p_2 \left(\frac{a}{a}\right)^3 - \left\{ 36 n_2^2 + \frac{1}{2} (2f_3 - f_1) p_2 \right\} \left(\frac{a}{a}\right)^4$$

$$- \frac{6}{5} f_2 n_2 \left(\frac{a}{a}\right)^5 - \left\{ \frac{44}{4900} f_2^2 - [illegible] + \frac{1}{2} (2f_3 - f_1) n_2 \right\} \left(\frac{a}{a}\right)^6 - \frac{3}{448} f_2 (2f_3 - f_1) \left(\frac{a}{a}\right)^7$$

$$- \left\{ \frac{11}{96^2} (2f_3 - f_1)^2 - \frac{75}{20} f_2 n_2 - [illegible] \right\} \left(\frac{a}{a}\right)^8 + \frac{11}{2800} f_2^2 \left(\frac{a}{a}\right)^9 + \left\{ \frac{1}{12800} f_2 (2f_3 - f_1) \right.$$

$$\frac{1}{1280}$$

$$\left. + \frac{3}{10} f_3 n_2 \right\} \left(\frac{a}{a}\right)^{10} + \frac{1}{[illegible] \times 16} f_2 [illegible] + \frac{[illegible]}{[illegible]} f_2^2 \left(\frac{a}{a}\right)^{12} + \frac{1}{800} f_2 f_3 \left(\frac{a}{a}\right)^{14} + \frac{3}{3200} f_3^2 \left(\frac{a}{a}\right)^{16}$$

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$$
\begin{aligned}
\frac{\frac{\tau_1}{R^3}}{\frac{\pi E}{2}\left(\frac{a}{R}\right)\left(\frac{t}{R}\right)} = {} & 0.35\frac{q}{p_0} + 0.175\,A\frac{q}{p_0} + 0.140\,B\frac{q}{p_0} + 0.04370833\,A^2 + 0.0583333\,C\frac{q}{p_0} + 0.07738571\,AB \\
& + 0.24000\,D\frac{q}{p_0} + 0.0365000\,B^2 + 0.03541667\,AC + 0.0145833\,E\frac{q}{p_0} + 0.03388889\,BC + 0.1552381\,AD \\
& + 0.20000\,F\frac{q}{p_0} + \cancel{0.00746111}\,C^2 + 0.15685143\,BD + 0.00989583\,AE + 0.036000\,G\frac{q}{p_0} + \\
& + \cancel{0.07174805\,CD} + 0.009734841\,BE + 0.14070401\,AF + 0.16969696\,H\frac{q}{p_0} + \cancel{0.1150718\,D^2} + \\
& + \cancel{0.0046875\,CE} + 0.1400020\,BF + 0.02041667\,AG + 0.005833333\,I\frac{q}{p_0} + \cancel{0.02148351}\,DE + \\
& + \cancel{0.06774871}\,CF + 0.0254615\,BG + 0.263703\,AH + \cancel{0.00070995\,E^2} + \cancel{0.3154616\,DF} \\
& + \cancel{0.0127405}\,CG + 0.1274459\,BH + 0.0443452\,AI - 0.100000\,J\frac{q}{p_0} + \cancel{0.0208333\,EF} + \\
& + \cancel{0.0517141}\,DG + \cancel{0.06262626}\,CH + 0.0045000\,BI + \cancel{0.1545714}\,F^2 + \cancel{0.0038541111\,EG} + \\
& + \cancel{0.0711487}\,DH + \cancel{0.00222222\,CI} - 0.07857143\,AJ - 0.050000\,K\frac{q}{p_0} + 0.0530588\,FG + \cancel{0.0115784\,EH} \\
& + \cancel{0.0104033}\,I - 0.0805042\,BJ + \cancel{0.0003444}\,G^2 + \cancel{0.2204010}\,FH + \cancel{0.000403}\,EI \\
& \qquad 0.0101359651 \\
& - \cancel{0.04007437}\,CJ - 0.04027777\,AK - \cancel{0.0364246}\,GH + 0.0100621\,FI - \cancel{0.1889581}\,DJ - 0.0415789\,BK \\
& + \cancel{0.1395226}\,H^2 + 0.00125833\,GI - \cancel{0.0023168}\,EJ - 0.0208333\,CK
\end{aligned}
$$

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$$+\left(0.01507215\,HI - 0.1918367\,FJ - 0.00984755\,GK\right) + \left(0.0001821338\,I^2 + 0.04003788\,GJ \quad \cancel{0.03616883511}\right.$$

$$\left. - 0.00672348\,EK\right) - 0.1418824\,HJ - 0.10131597\,FK - 0.00678521\,IJ - 0.01916667\,GK$$

$$+ 0.06365777\,J^2 - 0.00362177\,IK + 0.06836735\,JK + 0.01845238\,K^2 - 0.0993939394\,HK$$

$$+\Big\{ 104\,p_2^2 + 31.2\,p_2 n_2 + 0.52\,f_3 p_2 + 39.6\,n_2^2 + 0.21666667\,(2f_3 - f_1)p_2 + 1.4742857\,f_2 n_2$$

$$+ 0.01416837\,f_2^2 - 0.162500\,f_2 p_2 + 0.662500\,(2f_3 - f_1)n_2 + 0.01304563\,f_3(2f_3 - f_1) \quad 0.0053776054$$

$$+ 0.003059896\,(2f_3 - f_1)^2 - 0.542500 \text{ [illegible]} - 0.13000\,f_3 f_2 - 0.011155844\,f_2^2 - 0.0053225261\,f_2(2f_3 - f_1)$$

$$- 0.465\,f_3 n_2 - 0.00969505\,f_1 f_3 + 0.02460379\,f_3^2 - 0.004761905\,f_3(2f_3 - f_1) + 0.004484375\,f_2 f_3$$

$$+ 0.002086806\,f_3^2 \Big\}$$

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$$2f_3 - f_1 + 4f_1^2 - 8f_1f_3 - 4.5f_2^2$$

$$= -2.25000f_1 - 1.687500f_2 + 9f_1^2 + 6.750000f_1f_2 - 4.5000f_2^2$$

$$f_2(f_1 - 2f_3) = 2.250000f_1f_2 + 1.6875000f_2^2$$

$$12f_1f_2 + 16f_1f_3 - 16f_3^2 - 3f_2 - 4f_1^2$$

$$= -3f_2 - 20.25000f_1^2 - 18.37500f_1f_2 - 11.3906250f_2^2$$

$$4f_1f_3 + 6f_2f_3 - f_3 - 3f_1f_2 = 0.6250f_1 + 0.84375f_2 - 2.5000f_1^2 - 10.12500f_1f_2$$
$$- 5.062500f_2^2$$

$$f_2f_3 = -0.625000f_1f_2 - 0.84375f_2^2$$

$$32f_3^2 - 16f_1f_3 - 9f_2^2 = 16f_3(2f_3 - f_1) - 9f_2^2$$

$$= 16(0.625f_1 + 0.84375f_2)(2.2500f_1 + 1.68750f_2) - 9f_2^2$$

$$= 22.500f_1^2 + 47.2500f_1f_2 + 13.781250f_2^2$$

$$f_3^2 = 0.390625f_1^2 + 1.0546875f_1f_2 + 0.7119140625f_2^2$$

$$f_1(1 - 2f_1) = f_1 - 2f_1^2$$

$$f_2(1 - 4f_1) = f_2 - 4f_1f_2$$

To calculate the bending energy: <u>185</u>

$$k_1 = \frac{\partial^2 w}{\partial r^2} - \frac{\partial^2 w_0}{\partial r^2} = \frac{1}{R}\left\{ 2f_1\left[1-3\left(\frac{r}{a}\right)^2\right] + 3f_2\left[2-5\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 4f_3\left[3-7\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2 \right\}$$

$$k_2 = \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}$$

$$= \frac{1}{R}\left\{ 2f_1\left[1-\left(\frac{r}{a}\right)^2\right] + 3f_2\left[1-\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 4f_3\left[1-\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2 \right\}$$

$$k_1 + k_2 = \frac{1}{R}\left\{ 4f_1\left[1-2\left(\frac{r}{a}\right)^2\right] + 9f_2\left[1-2\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 16f_3\left[1-2\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2 \right\}$$

$$= \frac{1}{R}\left\{ 4f_1 + 9f_2\left(\frac{r}{a}\right) + 8(2f_3 - f_1)\left(\frac{r}{a}\right)^2 - 18f_2\left(\frac{r}{a}\right)^4 - 32f_3\left(\frac{r}{a}\right)^6 \right\}$$

$$\int_0^a (k_1+k_2)^2 r\,dr = \left(\frac{a}{R}\right)^2 \left\{ 8f_1^2 + 24f_1f_2 + \frac{81}{4}f_2^2 + 16f_1(2f_3-f_1) \right.$$

$$+ \frac{144}{5}f_2(2f_3-f_1) + \frac{32}{3}(2f_3-f_1)^2 - 24f_1f_2 - \frac{324}{7}f_2^2 - 32f_1f_3$$

$$- 36f_2(2f_3-f_1) - 64f_2f_3 + 32.4f_2^2 - 51.2f_3(2f_3-f_1) + 96f_2f_3$$

$$\left. + \frac{512}{7}f_3^2 \right\}$$

$$\int_0^a (k_1 k_2) r\,dr = \left(\frac{a}{R}\right)^2 \left\{ 2f_1^2 + 6f_1f_2 + \frac{9}{2}f_2^2 + 4f_1(2f_3-f_1) \right.$$

$$+ 6f_2(2f_3-f_1) + 2(2f_3-f_1)^2 - 6f_2f_1 \qquad - 9f_2^2 - 6f_2(2f_3-f_1)$$

$$\left. - 8f_1f_3 - 12f_2f_3 + 4.5f_2^2 - 8f_3(2f_3-f_1) + 12f_2f_3 + 8f_3^2 \right\}$$

$$\int_0^a (k_1+k_2)^2 r\,dr = \left(\frac{a}{R}\right)^2 \Big\{ 8 f_1^2 + 0 + 6.364286 f_2^2 + 16 f_1 (2f_3 - f_1)$$

$$- 72 f_2 (2f_3 - f_1) + \frac{32}{3} (2f_3 - f_1)^2 - 32 f_1 f_3 + 32 f_2 f_3 - 51.2 f_3 (2f_3 - f_1)$$

$$+ \frac{512}{7} f_3^2 \Big\}$$

$$\int_0^a (k_1 k_2) r\,dr = \left(\frac{a}{R}\right)^2 \Big\{ 2 f_1^2 + 0 + 0 + 4 f_1 (2f_3 - f_1) + 0$$

$$+ 2 (2f_3 - f_1)^2 - 8 f_1 f_3 + 0 - 8 f_3 (2f_3 - f_1) + 8 f_3^2 \Big\}$$

$$\frac{\tilde{\mathcal{C}}_2}{R^3} = \frac{1}{12}\left(\frac{t}{R}\right)^3 \frac{E\pi}{(1-\nu^2)} \left(\frac{a}{R}\right)^2 \Big\{ 28 f_1^2 + 6.364286 f_2^2 + 56 (2f_3 - f_1) f_1$$

$$- 72 f_2 (2f_3 - f_1) + 5.4666667 (2f_3 - f_1)^2 - 112 f_1 f_3 + 32 f_2 f_3$$

$$- 30.4 f_3 (2f_3 - f_1) + 52.34285714 f_3^2 \Big\}$$

$$\frac{\dfrac{\mathcal{C}_2}{R^3}}{\dfrac{\pi E}{2}\left(\dfrac{a}{R}\right)^2\left(\dfrac{t}{R}\right)^3} = 0.5128205 f_1^2 + 1.1656201 f_2^2 + 1.0256410 f_1 (2f_3 - f_1)$$

$$- 1.3186813 f_2 (2f_3 - f_1) + 1.0012210 (2f_3 - f_1)^2$$

$$- 2.0512421 f_1 f_3 + 5.8608059 f_2 f_3 - 5.5677656 f_3 (2f_3 - f_1)$$

$$+ 9.5866039 f_3^2$$

$$\frac{\mathcal{E}_1 / R^3}{\frac{1}{2}\frac{\sigma^2}{E}\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2} = 8 s_2^2 + 2(1+\nu)\left\{(r_0^2 + 2s_2^2) + 12 s_2 q_2 + 12 q_2^2\right\}$$

$$\frac{\delta\theta / R^3}{\frac{\sigma^2}{2E}\pi\left(\frac{t}{R}\right)\left(\frac{a}{R}\right)^2} = 4s_2 - 2(1+\nu) r_0$$

For the region outside the circle.

$$\widehat{rr} = \sigma\left[\frac{1}{2} + \frac{r_0}{\left(\frac{r}{a}\right)^2} + \cos 2\theta\left\{\frac{1}{2} - \frac{6q_2}{\left(\frac{r}{a}\right)^4} - \frac{4s_2}{\left(\frac{r}{a}\right)^2}\right\}\right]$$

$$\widehat{\theta\theta} = \sigma\left[\frac{1}{2} - \frac{r_0}{\left(\frac{r}{a}\right)^2} + \cos 2\theta\left\{\frac{6q_2}{\left(\frac{r}{a}\right)^4} - \frac{1}{2}\right\}\right]$$

$$\widehat{r\theta} = -\sigma \sin 2\theta\left\{\frac{1}{2} + \frac{6q_2}{\left(\frac{r}{a}\right)^4} + \frac{2s_2}{\left(\frac{r}{a}\right)^2}\right\}$$

$$\frac{u}{R} = \frac{\sigma}{E}\left(\frac{a}{R}\right)\left[\frac{1}{2}(1-\nu)\left(\frac{r}{a}\right) - (1+\nu)\frac{r_0}{\left(\frac{r}{a}\right)} + \cos 2\theta\left\{\frac{1}{2}(1+\nu)\left(\frac{r}{a}\right) + 2(1+\nu)\frac{q_2}{\left(\frac{r}{a}\right)^3} + \frac{4s_2}{\left(\frac{r}{a}\right)}\right\}\right]$$

$$\frac{v}{R} = \frac{\sigma}{E}\left(\frac{a}{R}\right)\left[2(1+\nu)\frac{q_2}{\left(\frac{r}{a}\right)^3} - \frac{1}{2}(1+\nu)\left(\frac{r}{a}\right)\right]\sin 2\theta$$

$$\frac{\mathscr{E}_2/R^3}{\frac{\pi E}{2}\left(\frac{a}{R}\right)^2\left(\frac{t}{R}\right)^3} = 0.5128205 f_1^2 + 1.1656201 f_2^2 - 1.0256410 f_1 (2.2500 f_1 + 1.6875 f_2)$$

$$+ 1.3186813 f_2 (2.2500 f_1 + 1.6875 f_2) + 1.0012210 (2.2500 f_1 + 1.6875 f_2)^2$$

$$+ 2.0512821 f_1 (0.625 f_1 + 0.84375 f_2) - 5.8608058 f_2 (0.625 f_1 + 0.84375 f_2)$$

$$- 5.5673156 (0.625 f_1 + 0.84375 f_2)(2.250 f_1 + 1.6875 f_2) + 9.5466034 (0.625 f_1 + 0.84375 f_2)^2$$

=

f_1^2	$f_1 f_2$	f_2^2
+0.5128205		+1.1656201
−2.3076923	−1.7307692	
	+2.9670329	+2.2252747
+5.0686813	+7.6030250	+2.8511332
+1.2820513	+1.7307693	
	−3.6630037	−4.9450550
−7.8296704	−16.4423078	−7.9275413
+3.7447671	+10.1108713	+6.8248381
+0.4709575	+0.5756148	+0.1942698

$$\mathscr{E}_2/R^3 = \frac{\pi E}{2}\left(\frac{a}{R}\right)^2\left(\frac{t}{R}\right)^3 \left\{ 0.4709575 f_1^2 + 0.5756148 f_1 f_2 + 0.1942698 f_2^2 \right\}$$

$g_0 - \frac{1}{7} =$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
−0.500000		+1.00000		
	−0.50000		+2.00000	
+0.562500	+0.421875	−2.25000	−1.6875	+1.125000
			−2.7000	−2.025000
	+0.750000	+1.6875	+1.53125	+0.94921875
				−1.28571429
−0.15625	−0.2109375	+0.625	+2.53125	+1.265625
			+0.2083333	+0.2812500
		−1.125	−2.3625	−0.6890625
		+0.22321429	+0.60267857	+0.40680104

$-0.0937500 f_1 - 0.03906250 f_2 + 0.16171429 f_1^2 + 0.12351190 f_1 f_2 + 0.02812500 f_2^2$

0.12351183

$h_0 =$ f_1	f_2	f_1^2	$f_1 f_2$	f_2^2	400
−0.1250		+0.2500			
	−0.150		+0.50000		
+0.16750	+0.146625	−0.25000	−0.52500	+0.37500	
			−0.96728231	−0.72321429	
	+0.09375	+0.530625	+0.742625	+0.3359[illegible]	
				−[illegible]	
−0.0625	−0.084375	+0.25	+1.0125	+0.50625	
			+0.07205195	+0.0986213	
		−0.[illegible]25	−0.187375	−0.2[illegible]	
		+0.09215525	+0.[illegible]25	+0.1272852	

[illegible]

$$h_0^2 = \gamma^2 \{ 0.0001373291 f_1^4 + 0.0002878563 f_1^3 f_2 + 0.0002324539 f_1^2 f_2^2 + 0.0000155337 f_1 f_2^3 + 0.0000121244 f_2^4 \}$$

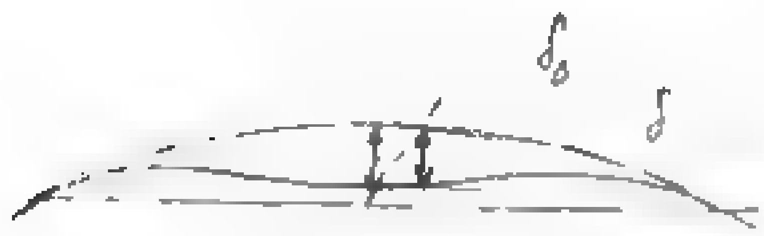
$$h_2^2 = \gamma^2 \{ 0.0000203455 f_1^2 + 0.00[illegible] f_1 f_2 + 0.0000[illegible] \}$$

$$\mathcal{E}_3/R^3 = \frac{1}{2}\frac{\sigma^2}{E}\pi\left(\frac{a}{R}\right)^2\left(\frac{t}{R}\right)\Big\{ 0.0003570557 f_1^4 + 0.0007484264 f_1^3 f_2$$

$$+ 0.0006043801 f_1^2 f_2^2 + 0.0002223868 f_1 f_2^3 + 0.0000315234 f_2^4$$

$$+ 0.0000528983 f_1^2 + 0.0000612144 f_1 f_2 + 0.0000177094 f_2^2 \Big\}\eta^2$$

$$\mathcal{E}_2/R^2 = \frac{1}{2}\frac{\sigma^2}{E}\pi\left(\frac{a}{R}\right)^2\left(\frac{t}{R}\right)\eta\Big\{ -0.03046875 f_1^2 - 0.03193286 f_1 f_2 - 0.00905323 f_2^2 \Big\}$$



$$\frac{\delta}{t_0} = f_1 + f_2 + f_3$$

$$= 0.375 f_1 + 0.15625 f_2$$

The [illegible] p. 393 can be collecte[illegible] ..

$$\Big\{ 10.4 f_2^2 + 31.2 f_2 n_2 + 0.3225 f_2 f_2 + 39.6 n_2^2 - 0.21666667 f_2 (2.250 f_1 + 1.6875 f_2)$$

$$+ 0.92678857 f_2 n_2 + 0.00547291 f_2^2 - 0.6625000 (2.2500 f_1 + 1.6875 f_2) n_2$$

$$- 0.00766803 f_2 (2.2500 f_1 + 1.6875 f_2) + 0.003059896 (2.2500 f_1 + 1.6875 f_2)^2$$

$$+ 0.13000 f_2 (0.625 f_1 + 0.84375 f_2) + 0.465 n_2 (0.625 f_1 + 0.84375 f_2)$$

$$+ 0.00521067 f_2 (0.625 f_1 + 0.84375 f_2) - 0.004761905 (0.625 f_1 + 0.84375 f_2)$$

$$(2.25 f_1 + 1.6875 f_2) + 0.00208680 (0.625 f_1 + 0.84375 f_2)^2 \Big\}$$

$= 0.6500000\,t$

$$+\eta\Big\{\big(\overset{0.0609378}{0.0058594}\,f_1 + \overset{0.0253906}{0.0024414\,0}\,f_2\big)\times 10.4 - 0.1625000\,f_1 - 0.089375\,f_2$$

$$+ 0.121875\,f_1 + 0.0914062\,f_2 - 0.0203125\,f_1 - 0.02742187\,f_2\Big\}$$

$\rightarrow$ 0 + $\cdot$ 0

$+\eta^2\Big\{$

f_1	f_2	f_1^2	f_1f_2	f_2^2
		+0.00142824	+0.00119019	+0.00024795
		−0.00761722	−0.00317382	
			−0.00416947	−0.00174560
				−0.00160260
		+0.0171875		
			+0.01930804	
				+0.00542271
		−0.03105469	−0.02329102	
			−0.01725307	−0.01273980
		+0.01549072	+0.02323609	+0.00871353
		+0.00605469	+0.00817383	
			+0.00325617	+0.00439650
		−0.0011205	−0.00687765	−0.00404448
		−0.00683343	−0.01351121	−0.00577621
		$-0.00534419\,f_1^2$	$-0.00625377\,f_1f_2$	$-0.00149772\,f_2^2$
		+0.00036874	+0.0037879	+0.00010068

Independent check

η^2	f_1^2	$f_1 f_2$	f_2^2
	+0.00142824	+0.00119019	+0.00024295
	−0.00761722	−0.00357362	
		−0.00418947	−0.00147760
	+0.01711625		
		+0.01930804	
			+0.00542191
	−0.00985104	−0.01064295	−0.00244103
	−0.00077874	−0.00211321	−0.00143358
	+0.00036874	+0.00037428	+0.00101068

$A = f_1 \; 2f_1^2$

$B = f_2 - 4f_1 f_2$

$C = -2.25000 f_1 - 1.6875 f_2 + 9 f_1^2 + 6.75000 f_1 f_2 - 4.50000 f_2^2$

$D = 3.25000 f_1 f_2 + 1.68750 f_2^2$

$E = -3 f_2 - 30.25000 f_1^2 - 18.37500 f_1 f_2 - 11.5700250 f_2^2$

$F = f_2^2$

$G = 0.625 f_1 + 0.14375 f_2 - 2.5000 f_1^2 - 10.12500 f_1 f_2 - 5.0625000 f_2^2$

$H = -0.6250 f_1 f_2 - 0.14375 f_2^2 = J$

$I = 22.500 f_1^2 + 47.2500 f_1 f_2 + 13.78125 f_2^2$

$K = 0.390625 f_1^2 + 1.0546875 f_1 f_2 + 0.7119140625 f_2^2$

$\eta\%_0 = 1 + \eta \{ -0.09375 f_1 - 0.0[illegible] f_2 + [illegible] f_1^2 - [illegible] f_1 f_2$
$+ 0.0781250 f_2^2 \}$

$\eta\%_0^2 = 1 + \eta \{ -0.1875 f_1 - 0.078125 f_2 + 0.32142858 f_1^2 + 0.24702380 f_1 f_2 + 0.0562500 f_2^2 \}$

$+ \eta^2 \{ +0.0087890625 f_1^2 + 0.0073242187 5 f_1 f_2 + 0.0015258789 1 f_2^2$
$- 0.03013393 f_1^3 - 0.03571429 f_1^2 f_2 - 0.01493380 f_1 f_2^2 - 0.00219722 f_2^3$
$+ 0.02583908 f_1^4 + 0.03970025 f_1^3 f_2 + 0.02429537 f_1^2 f_2^2 + 0.00694754 f_1 f_2^3$
$+ 0.000791016 f_2^4 \}$

$\times\ 0.35$!!!

$$\eta^2 q_0 \Big[0.175A + 0.140B + 0.05833333C + 0.24D + 0.01458333E + 0.2F + 0.035G + 0.\overset{0.06969696969}{16969696969\!6}H + 0.00583333I - 0.1J - 0.05K \Big]$$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
0.175		−0.35		
	0.14		−0.56	
−0.13125	−0.0984375	+0.525	+0.39375	−0.2625
			+0.54	+0.405
	−0.04375	−0.2953125	−0.26796875	−0.16611328
				+0.2
+0.021875	+0.02953125	−0.0875	−0.354375	−0.1771875
			−0.04356060 6	−0.05880682
		+0.13125	+0.275625	+0.06039 0625
		−0.01953125	−0.052734375	−0.025595703125

$$\left(+0.065625 f_1 + 0.02734375 f_2 - 0.09609375 f_1^2 - 0.06926374 f_1 f_2 - 0.01481267 f_2^2\right)\eta$$

$$\eta q_0 = 1 + \eta\left\{-0.09375 f_1 - 0.0390625 f_2 + 0.16071429 f_1^2 + 0.12351190 f_1 f_2 + 0.02812500 f_2^2\right\}$$

$$- \eta\left\{0.065625 f_1 + 0.02734375 f_2 - 0.09609375 f_1^2 - 0.06926374 f_1 f_2 - 0.01481267 f_2^2\right\}$$

$$- \eta^2\Big\{0.00615234 f_1^2 + 0.00512695 f_1 f_2 + 0.01068 12 f_2^2 - 0.01955561 f_1^3 - \overset{-0.0227474714}{\boxed{-0.01631860}} f_1^2 f_2 - 0.00931728 f_1 f_2^2 - 0.00134766 f_2^3 + 0.01544364 f_1^4 + 0.02300039 f_1^3 f_2 + 0.01363814 f_1^2 f_2^2 + 0.00377758 f_1 f_2^3 + 0.00041161 f_2^4\Big\}$$

$$\eta^{4}A\Big[0.04270833A + 0.0778571B + 0.03541667C + 0.1557381D + 0.09895833E \quad \underline{405}$$

$$\overset{0.0477689}{} \quad + 0.14090909F + 0.02541667G + 0.1263403H + 0.0443452I - 0.02857143J$$

$$- 0.04027777K\Big]$$

f_1	f_2	f_1^2	f_1f_2	f_2^2
+0.04270833		−0.08541667		
	+0.0778571		−0.3114284	
−0.0296875	−0.059765625	+0.31875	+0.2390625	−0.159375
			+0.3492857	+0.2619643
	−0.0296875	−0.200390625	−0.1416559	−0.1122197
				+0.1409091
+0.0158854	+0.0244453	−0.0635417	−0.0553385	−0.1286219
			−0.0298556	−0.0403050
		+0.0777767	+0.2015311	+0.2611132
		+0.0997317	+0.2094316	+0.0610851
		−0.0157335	−0.0424805	−0.0286743

$$\{-0.0210938f_1 + 0.0091493f_2 + 0.0533992f_1^2 - 0.0251594f_1f_2 - 0.0057819f_2^2\}\eta$$

$$\qquad\qquad +0.0534492 \qquad -0.0250649 \qquad -0.0057587$$

$$-(-f_1 + 2f_1^2)\eta^2$$

$$= -\eta^2\Big\{+0.0210938f_1^2 - 0.0091493f_1f_2 \overset{-0.0956318}{-0.0955868}f_1^3 \overset{+0.0447635}{+0.0448580}f_1^2f_2$$

$$+ 0.0057869f_1f_2^2 + 0.1067784f_1^4 - 0.0503188f_1^3f_2 - 0.0115738f_1^2f_2^2\Big\}$$

$$+0.0057587 \qquad +0.1068584 \qquad -0.0501298 \qquad -0.0115174$$

$\eta^2 B[0.0365 B + 0.03388889 C + 0.150857143 D + 0.009734848 E + 0.14 F$ **406**

$+ 0.0254615 G + \overset{0.0469417}{\cancel{0.72744459}} H + 0.0045 I - 0.0805042 J - 0.0415719 K]$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
	$+0.0365$		-0.146	
-0.07625	-0.0571875	$+0.305$	$+0.22875$	-0.1525
			$+0.3394216$	$+0.254[illegible]214$
	-0.0292045	-0.1971307	-0.1788728	-0.1108860
				$+0.14$
$+0.0159134$	$+0.0214831$	-0.0636538	-0.2577927	-0.1288988
			-0.0293386	-0.0396071
		$+0.10125$	$+0.212625$	$+0.062015625$
		-0.0112418	-0.0438527	-0.0176006

$-0.0603[illegible]6 f_1 - 0.0284089 f_2 + 0.1272237 f_1^2 + 0.[illegible]9316 f_1 f_2 - 0.00[illegible] f_2^2$

$0 \quad + \quad f_2 \quad + \quad 0 \quad - 4 \quad f_1 f_2 \quad + \quad 0$

$\eta^2 \{ -0.0[illegible]6 f_1 f_2 - 0.0284[illegible] f_2^2 + 0.[illegible] f_1^2 f_2 + 0.231[illegible] f_1 f_2^2$

$- 0.0049055 f_2^3 - 0.5168948 f_1^3 f_2 - 0.4997472 f_1^2 f_2^2 + 0.0196220 f_1 f_2^3 \}$

$C[0.007986111\,C + 0.07194405\,D + 0.0046875\,E + 0.067948718\,F$ <u>407</u>

$+ 0.0124405\,G + 0.02254689\,H + 0.00222222\,I - 0.070833333\,K]$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
-0.0179687	-0.0134766	$+0.071875$	$+0.05390625$	-0.0559375
			$+0.1618831$	$+0.1214123$
	-0.0140625	-0.0949219	-0.0861328	-0.0533936
				$+0.0679487$
$+0.0077753$	$+0.0104967$	-0.0311013	-0.1359601	-0.0629806
			-0.0140918	-0.0170939
		$+0.05$	$+0.105$	$+0.030625$
		-0.0081360	-0.0219726	-0.0144315

$-0.0101934\,f_1 - 0.0170424\,f_2 - 0.0122862\,f_1^2 + 0.0726390\,f_1 f_2 + 0.038195\,f_2^2$

$-2.25\,f_1 - 1.6875\,f_2 + 9\,f_1^2 + 6.25\,f_1 f_2 - 4.5\,f_2^2$

$= r_1^2\{0.0227352\,f_1^2 + 0.0553468\,f_1 f_2 - 0.083791\,f_2^2 -$

$- 0.0640967\,f_1^3 \overset{-0.3643791}{-0.0475788}\,f_1^2 f_2 - 0.2678263\,f_1 f_2^2 + 0.0196204 \ldots$

$- 0.1105758\,f_1^4 + 0.5707562\,f_1^3 f_2 + 0.6499094\,f_1^2 f_2^2 - 0.0985624\,f_1 f_2^3$

$- 0.1521878\,f_2^4\}$

$$D\left[0.1635918\,D + 0.021483516\,E + 0.3134694\,F + 0.05771429\,G + 0.1029900\,H + 0.0104033\,I - 0.00987755\,K\right]$$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
			+0.3680816	+0.2760612
	-0.0644505	-0.4250412	-0.3947596	-0.2447107
				+0.3134694
+0.0360714	+0.0486964	-0.1442457	-0.5843572	-0.2921786
			-0.0643688	-0.0868978
		+0.234075	+0.4915575	+0.1433709
		-0.0038584	-0.0104177	-0.0070320

$$+0.0360714\,f_1 - 0.0157541\,f_2 - 0.3491103\,f_1^2 - 0.1942642\,f_1 f_2 + 0.1020824\,f_2^2$$

$$+ 2.25\; f_1 f_2 + 1.1235\; f_2^2$$

$$7^2\Big\{+0.0811607\,f_1^2 f_2 + 0.0254238\,f_1 f_2^2 - 0.0265850\,f_2^3 - 0.7854982\,f_1^3 f_2 - 1.0262181\,f_1^2 f_2^2 - 0.0981354\,f_1 f_2^3 + 0.1722641\,f_2^4\Big\}$$

$$E\Big[\, 0.00070995\, E + 0.0208333\, F + 0.0038541667\, G + 0.0067805\, H + 0.00070023\, I - 0.00672348\, K \,\Big]$$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
	-0.0021299	-0.0143765	-0.0130453	-0.00808677
				$+0.0208333$
$+0.0034089$	$+0.0032590$	-0.0096354	-0.0390234	-0.0195117
			-0.0042378	-0.0057210
		$+0.0157552$	$+0.0330859$	$+0.0096509\,45$ $+0.0097131$
		-0.0026264	-0.0070912	-0.0047865

$$+0.0034089 f_1 + 0.0011221 f_2 - 0.0108831 f_1^2 - 0.0303118 f_1 f_2 \; \overset{-0.0076226}{-0.0075596} f_2^2$$

$$-3 \; f_2 - 20.25 \; f_1^2 - 18.375 \; f_1 f_2 - 11.390625 \; f_2^2$$

$$\eta^2 \Big\{ -0.0072267 f_1 f_2 - 0.0033663 f_2^2 - 0.0487802 f_1^3 - 0.0343368 f_1^2 f_2 + 0.0426779 f_1 f_2^2 + \overset{+0.0100864}{0.0098974} f_2^3 + 0.2203828 f_1^4 + 0.8137909 f_1^3 f_2 + \overset{+0.8353022}{0.8340265} f_1^2 f_2^2 + \overset{+0.4853356}{0.4841780} f_1 f_2^3 + \underset{+0.0868262}{0.0861086} f_2^4 \Big\}$$

$$F\Big[\,0.1535714\,F + 0.0570588\,G + \overset{0.09907239}{0.101092626}\,H + 0.010438597\,I - 0.101242236\,K\,\Big]$$

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f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
				+0.1535714
+0.0356618	+0.0481434	−0.1426470	−0.5777204	−0.2888602
			−0.0619202	−0.0835923
			−0.0631829	−0.0852969
		+0.2348684	+0.4932237	+0.1438569
		−0.0395477	−0.1067289	−0.0720258

$$+0.0356618\,f_1 f_2^2 + 0.0481434\,f_2^3 + 0.0526727\,f_1^2 f_2^2 - 0.2544585\,f_1 f_2^3 - 0.1488146\,f_2^4$$

−0.2531958 −0.1471 0000

$$G\left[0.00531944\,G + 0.01824903\,H + 0.00195833\,I - 0.01916667\,K\right]$$ 412

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
+0.0033247	+0.0044883	-0.0132986	-0.0534593	-0.0269297
			-0.0114056	-0.0153976
		+0.0440625	+0.09253125	+0.02698828
		-0.0074870	-0.0202148	-0.0136450

$$+0.0033247\,f_1 + 0.0044883\,f_2 + 0.0232769\,f_1^2 + 0.0070516\,f_1 f_2 - 0.0289840\,f_2^2$$

$$+0.625\,f_1 + 0.84375\,f_2 - 2.5\,f_1^2 - 10.125\,f_1 f_2 - 5.0625\,f_2^2$$

$$\eta^3\Big\{0.0020779\,f_1^2 + 0.0058104\,f_1 f_2 + 0.0102530\,f_2^2 +$$
$$+ 0.0062363\,f_1^3 - 0.0268262\,f_1^2 f_2 - 0.0344705\,f_1 f_2^2 - 0.0471773\,f_2^3 -$$
$$- 0.0081923\,f_1^4 - 0.0533076\,f_1^3 f_2 - 0.1617252\,f_1^2 f_2^2 + 0.2572143\,f_1 f_2^3$$
$$+ 0.1467215\,f_2^4\Big\}$$

$$H\left[0.0163730\,H + 0.0032864\,4\,I - 0.0310265\,9\,K\right]$$

f_1^2	$f_1 f_2$	f_2^2
	-0.0102331	-0.0138147
$+0.0739449$	$+0.1552843$	$+0.0452913$
-0.0121198	-0.0327234	-0.0220883
$+0.0618251\,f_1^2$	$+0.1123278\,f_1 f_2$	$+0.0093883\,f_2^2$

$$\times \quad 0 \qquad -0.625\ f_1 f_2 - 0.84375\ f_2^2$$

$$\eta^2\left\{-0.0386407\,f_1^3 f_2 - 0.1023598\,f_1^2 f_2^2 - 0.1564473\,f_1 f_2^3 - 0.0079214\,f_2^4\right\}$$

$$K\left[0.0184523\,8\,K - 0.0036217\,9\,I\right]$$

f_1^2	$f_1 f_2$	f_2^2
$+0.0072080$	$+0.0194615$	$+0.0131365$
-0.0814903	-0.1711296	-0.0499128
$-0.0742823\,f_1^2$	$-0.1516681\,f_1 f_2$	$-0.0367763\,f_2^2$

$$\times)\quad 0.390625\ f_1^2 + 1.0546875\,f_1 f_2 + 0.7119140625\,f_2^2$$

$$\eta^2\left\{-0.0290165\,f_1^4 - 0.1375900\,f_1^3 f_2 - 0.2272108\,f_1^2 f_2^2 - 0.1467622\,f_1 f_2^3 - 0.0261816\,f_2^4\right\}$$

$$0.0001\, I^2 = \eta^2 \left\{ 0.225\, f_1^2 + 0.4725\, f_1 f_2 + 0.1378125\, f_2^2 \right\}^2$$

$$= \eta^2 \left\{ 0.050625\, f_1^4 + 0.212625\, f_1^3 f_2 + 0.28527187\,5\, f_1^2 f_2^2 + 0.1302328\, f_1 f_2^3 + 0.0189923\, f_2^4 \right\}$$

$$\eta^2\, 0.0001821338\, I^2 = \eta^2 \left\{ 0.0932072\, f_1^4 + 0.3872620\, f_1^3 f_2 + 0.7[illegible]\, f_1^2 f_2^2 + 0.2371979\, f_1 f_2^3 + 0.0345914\, f_2^4 \right\}$$

Terms with coefficient η

f_1^2	$f_1 f_2$	f_2^2
+0.1125000	+0.0864583	+0.0196875
−0.0960938	−0.0692637	−0.0148127

$$\eta \left(+0.0164062\, f_1^2 - 0.0171946\, f_1 f_2 + 0.0048748\, f_2^2 \right) \quad [illegible]$$
$$\eta \left(+0.0304688\, f_1^2 + 0.0319329\, f_1 f_2 + 0.0091032\, f_2^2 \right) \quad [illegible]$$

$$\eta \left(+0.0468750\, f_1^2 + 0.0491275\, f_1 f_2 + 0.0139780\, f_2^2 \right)$$

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f^2

f_1^2	$f_1 f_2$	f_2^2	f_1^3	$f_1^2 f_2$	$f_1 f_2^2$	f_2^3	
+0.0003687(4) −0.0053442	+0.0037829 −0.0062538	+0.0001006(8) −0.0014977					
+0.0030762	+0.0025635	+0.0005341	−0.015769	−0.012500	−0.0052230	−0.0007690	
−0.0061523	−0.0051270	−0.0010681	+0.01[illegible]57	+0.0227471 +0.0163186	+0.0093173	+0.0013477	
−0.0210938	+0.0018713	0	+0.0095137 +0.0[illegible]	−0.0447635 −0.0448580	−0.0057587 −0.0058667	0	
0	−0.0103366	−0.0027469		+0.3205701	+0.2385724	−0.0049055	
+0.0017952	+0.0055468	+0.0047591	−0.0[illegible]	−0.0245761 −0.0995768	−0.2648263	+0.0796204	
0	0	0	0	+0.0811607	+0.0254238	−0.0265850	
0	−0.0072267	−0.0033663	−0.0[illegible]	−0.0[illegible]568	+0.0428779	+0.0100864 +0.0079847	
0	0	0	0	0	+0.0356618	+0.0481434	
+0.0020779	+0.0056104	+0.0037870	+0.0[illegible]363	−0.0178362	−0.0344405	−0.0421773	
+0.0012114 $-0.0015010 f_1^2$	+0.0046676 $-0.0053711 f_1 f_2$	+0.0033158 $-0.0012608 f_2^2$	−0.0020000 −0.003[illegible]	−0.0029347 $+0.0[illegible]3965 f_1^2 f_2$	−0.0013953 $-0.0014235 f_1 f_2^2$	$-0.0002389 f_2^3$	ε_1

$+0.0000529 f_1^2 + 0.0000112 f_1 f_2 + 0.0000177 f_2^2$ $f_1 \ldots \varepsilon_3$

$+0.0012648 f_1^2 + 0.0047268 f_1 f_2 + 0.0033935 f_2^2$

f_1^4	$f_1^3 f_2$	$f_1^2 f_2^2$	$f_1 f_2^3$	f_2^4
+0.0070402	+0.0013895	+0.0085034	+0.0024316	+0.0002769
-0.0154436	-0.0730004	-0.0131381	-0.0037776	-0.0004166
-0.1065884 -0.1067984	+0.0509297 +0.0503168	+0.0115137 +0.0115738	0	0
0	-0.5168948	-0.4197[illegible]	+0.0796220	0
-0.1105758	+0.5707562	+0.4999994	-0.0975624	-0.1521878
0	-0.7854982	-1.0262181	-0.0781354	+0.1722641
+0.2203828	+0.1137909	+0.8353022 +0.8340265	+0.4853356 +0.4841780	+0.0862262 +0.0861086
0	0	+0.0526737	-0.2531458 -0.2544565	-0.1471030 -0.1466046
-0.0581723	-0.1533076	-0.1164[illegible]	+0.1571645	+0.1462315
0	-0.0341407	-0.113[illegible]16	-0.1001443	-0.0079214
[illegible]	-0.1315900	-0.0342[illegible]	-0.11[illegible]22	-0.0261816
+0.0722052	+0.3472690	+0.5157[illegible]	+0.2311979	+0.0345914

+0.0015116 f_1^4 +0.0009023 $f_1^3 f_2$ + 0.2715410 [illegible] + 0.0309[illegible]37 $f_1 f_2^3$ + 0.1063185 f_2^4

+0.[illegible] f_1^4 + 0.[illegible] $f_1^3 f_2$ + 0.[illegible] [illegible] 0.1041605 f_2^4 — from $\mathcal{E}_1$

+0.[illegible]3521 f_1^4 + 0.[illegible] $f_1^3 f_2$ + 0.0[illegible] [illegible] + 0.00[illegible] $f_1 f_2^3$ + 0.0020315 f_2^4 — from $\mathcal{E}_3$

(+0.[illegible]8687 f_1^4 + 0.0816507 $f_1^3 f_2$ + 0.2221462 $f_1^2 f_2^2$ + 0.301496[illegible] $f_1 f_2^3$ + 0.1063500 f_2^4) η^2

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$$\frac{M}{R^3} = \frac{1}{2}\frac{a^3}{E^2}\pi\left(\frac{\epsilon}{R}\right)\Big[\eta\left(0.6709575 f_1^2 + 0.5756148 f_1 f_2 + 0.1942698 f_2^2\right)\frac{1}{K^2}$$

$$-\eta^2\left(0.0468750 f_1^2 + 0.0491275 f_1 f_2 + 0.0139280 f_2^2\right)$$

$$+\eta^3\left(0.0012648 f_1^2 + 0.0047266 f_1 f_2 + 0.0033935 f_2^2 - 0.0020000 f_1^3 - 0.0028347 f_1^2 f_2\right.$$

$$-0.0013953 f_1 f_2^2 - 0.0002569 f_2^3 + 0.0018687 f_1^4 + 0.0816507 f_1^3 f_2 + 0.2421462 f_1^2 f_2^2$$

$$\left.+ 0.3014961 f_1 f_2^3 + 0.1063500 f_2^4\right)\Big]$$

If $f_2 = 0$, the conditions for equilibrium are

$$\frac{0.6709575}{K^2} - 0.0937500\eta + \eta^2(0.0037944 - 0.0060000 f_1 + 0.0056061 f_1^2) = 0$$

$$\frac{0.9419150}{K^2} - 0.[illegible]\eta + \eta^2(0.0005296 - 0.0050000 f_1 + 0.0074448 f_1^2) = 0$$

This set of equations can be put into the form

$$\eta^2 + A\eta + \frac{B}{K^2} = 0$$

$$\eta^2 + C\eta + \frac{D}{K^2} = 0$$

The resultant will be

$$\begin{vmatrix} C & \frac{D}{K^2} & 0 \\ 1 & A & \frac{B}{K^2} \\ 1 & C & \frac{D}{K^2} \end{vmatrix} - \begin{vmatrix} A & \frac{B}{K^2} & 0 \\ 1 & A & \frac{B}{K^2} \\ 1 & C & \frac{D}{K^2} \end{vmatrix} = 0$$

If $f_2=0$, the condition for equilibrium is <u>418</u>

$$\frac{0.9419150}{K^2} - 0.0937500\,\eta + \eta^2\left(0.0022296 - 0.0060000 f_1 + 0.0074748 f_1^2\right) = 0$$

where $\eta = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2$, hence $\frac{\delta}{R} = f_1 \cdot \frac{1}{2}\left(\frac{a}{R}\right)^2$

$$\frac{\delta}{t} = \mu = \frac{1}{2}\left(\frac{a}{R}\right)^2 \frac{f_1}{\left(\frac{t}{R}\right)}$$

or $$f_1 = \frac{\mu\left(\frac{t}{R}\right)}{\frac{1}{2}\left(\frac{a}{R}\right)^2}$$

Thus $$\frac{0.9419150}{K^2} - 0.0937500\left(\frac{E}{\sigma}\right)\left(\frac{a}{R}\right)^2 + \left(\frac{E}{\sigma}\right)^2\left(\frac{a}{R}\right)^4\left(0.0022296 - 0.0060000\,\frac{\mu\left(\frac{t}{R}\right)}{\frac{1}{2}\left(\frac{a}{R}\right)^2}\right.$$
$$\left. + 0.0074748\,\frac{\mu^2\left(\frac{t}{R}\right)^2}{\frac{1}{4}\left(\frac{a}{R}\right)^4}\right) = 0$$

$$\frac{0.9419150}{K^2} - \frac{0.0937500}{K}\,\gamma^2 + \frac{1}{K^2}\left(0.0022296\,\gamma^4 - 0.012000\,\mu\gamma^2\right.$$
$$\left. + 0.0298992\,\mu^2\right) = 0.$$

Thus
$$0.09375\,K\gamma^2 = 0.0022296\,\gamma^4 - 0.012000\,\mu\gamma^2 + \left(0.941915 + 0.0298992\,\mu^2\right)$$

$$K = 0.026982\,\gamma^2 - 0.12800\,\mu + \frac{10.047 + 0.31892\,\mu^2}{\gamma^2}$$

$$\gamma^2 = \sqrt{\frac{10.047 + 0.31892\mu^2}{0.026982}}$$

$$K = 2\sqrt{0.026982\,(10.047 + 0.31892\,\mu^2)} - 0.12800\mu$$

$$= 0.3286\sqrt{10.047 + 0.31892\,\mu^2} - 0.12800\mu$$

$\mu = 0.5$, $K = 0.1856\sqrt{31.503 + \mu^2} - 0.12800\mu$

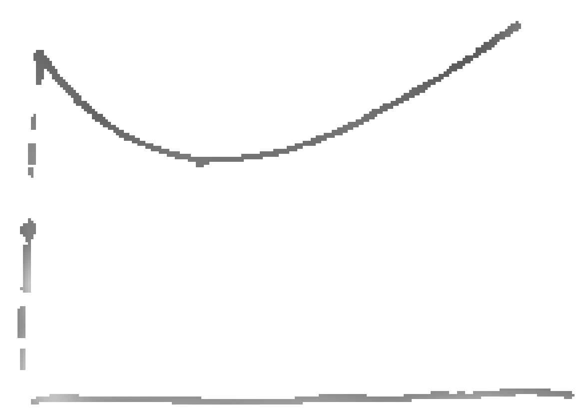
$= 1.046 -$

$\mu = 2$, $K = 0.850$

$\mu = 4$, $K = 0.768$

$\mu = 6$, $K = 0.758$

$\mu = 7$, $K = 0.817$



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$$\frac{U}{\ldots} = \frac{1}{3}\frac{\alpha^2}{E}\ldots\Big\{ f_1^2\left(0.4709525 + 0.5756148\rho + 0.1742678\rho^2\right)\frac{1}{K^2}$$

$$- \eta f_1^2 \left(0.0461750 + 0.0491275\rho + 0.0\ldots\rho^2\right) + \eta^2 f_1^2 \Big[0.012648 + 0.0061288\rho + 0.0023935\rho^2$$

$$- f_1\left(0.0030000 + 0.0028347\rho + 0.00\ldots\rho^2 + 0.0002387\rho^3\right) + f_1^2\left(0.0011617 + 0.0111507\rho + 0.2721462\rho^2\right.$$

$$\left.+ 0.3014761\rho^3 + 0.1063500\rho^4\right)\Big]\Big\}$$

$$\rho = \frac{f_2}{f_1}$$

$$\left(0.9419150 + 1.1512296\rho + 0.3485396\rho^2\right)\frac{1}{K^2} - \eta\left(0.0937500 + 0.078255\rho + 0.0276560\rho^2\right)$$

$$+ \eta^2\Big[0.0025296 + 0.0074576\rho + 0.0067870\rho^2 - f_1\left(0.006000 + 0.0085041\rho + 0.0041659\rho^2 + 0.0002167\rho^3\right)$$

$$+ f_1^2\left(0.0074748 + 0.3260988\rho + 1.0815548\rho^2 + 1.2057844\rho^3 + 0.425400\rho^4\right)\Big] = 0$$

$$\left(0.5756148 + 0.3485396\rho^2\right)\frac{1}{K^2} - \eta\left(0.0491275 + 0.0\ldots\rho\right)$$

$$+ \eta^2\Big[0.0061288 + 0.0067870\rho - f_1\left(0.0028347 + 0.0027906\rho + 0.0007161\rho^2\right)$$

$$+ f_1^2\left(0.0111507 + 0.5442924\rho + 0.1044883\rho^2 + 0.425400\rho^3\right)\Big] = 0$$

$$\frac{1}{K^2}(0.9419150 + 0.5756148\rho) - \frac{1}{K}\gamma^2(0.0491275 + 0.0\ldots1275\rho)$$

$$+ \frac{1}{K^2}\gamma^4\Big[0.0025296 + 0.0012066\rho - \frac{\mu_1}{\gamma^2}(0.0120000 + 0.0113388\rho + 0.00\ldots706\rho^2)$$

$$+ \frac{\mu_1^2}{\gamma^4}(0.0298992 + 0.9418084\rho + 2.1771696\rho^2 + 1.8059844\rho^3)\Big] = 0$$

$$\mu_1 = \frac{t}{2}\frac{(\frac{\rho}{n})^2}{(\frac{\rho}{n})}\delta_1 = \frac{\delta_1}{t}$$

$$\frac{1}{K^2}(0.5156148 + 0.3885396\rho) - \frac{1}{K}\gamma^2(0.0491275 + 0.0278565\rho)$$

$$+ \frac{1}{K^2}\gamma^4\Big[0.0\ldots688 + 0.0067770\rho - \frac{\mu_1}{\gamma^2}(0.0056674 + 0.0055812\rho + 0.0014334\rho^2)$$

$$+ \frac{\mu_1^2}{\gamma^4}(0.3266048 + 2.144\ldots\rho + 3.6179532\rho^2 + 1.701600\rho^3) = 0$$

$$(0.0298712 + 0.9418084\rho + 2.1771696\rho^2 + 1.8059844\rho^3)\mu_1^2 - \gamma^2(0.012 + 0.0113388\rho + 0.0027706\rho^2)\mu_1$$

$$+ \Big[0.941915 + 0.5756148\rho + (0.0025296 + 0.0012066\rho)\gamma^4 - K\gamma^2(0.0491250 + 0.0491275\rho)\Big] = 0.$$

$$(0.3266048 + 2.144\ldots76\rho + 3.6179532\rho^2 + 1.70\ldots\rho^3)\mu_1^2 - \gamma^2(0.0056674 + 0.0055812\rho + 0.0014334\rho^2)\mu_1$$

$$+ \Big[0.5156148 + 0.3885396\rho + (0.00\ldots + 0.00\ldots770\rho)\gamma^4 - K\gamma^2(0.0491275 + 0.0278565\rho)\Big] = 0$$

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$$\mu = \left(\frac{d}{t}\right) = 0.375\,\mu_1 + 0.15625\,\mu_2 = \mu_1\,(0.375 + 0.15625\,\rho)$$

$$\frac{\mu/\mu_1 - 0.375}{0.15625} = \rho = 6.4000\,\mu/\mu_1 - 2.08000$$

$$\mu_1^3\Big[0.0298992 + 0.9790084\,(6.4000\,\mu/\mu_1 - 2.08000) + 2.1721696\,(40.96\,\mu^2/\mu_1^2 - 26.624\,\mu/\mu_1 + 4.3264)$$
$$+ 1.2059644\,(262.144\,\mu^3/\mu_1^3 - 255.5904\,\mu^2/\mu_1^2 + 83.06688\,\mu/\mu_1 - 8.998912)\Big]$$

$$= \left[-3.4413432\,\mu_1^3 + 48.483172\,\mu_1^2\mu - 21.906117\,\mu_1\mu^2 + 3.614157\,\mu^3\right]$$

$$\mu_1^2\Big[0.012 + 0.011328\,(6.4000\,\mu/\mu_1 - 2.0800) + 0.002[illegible]\,(40.96\,\mu^2/\mu_1^2 - 26.624\,\mu/\mu_1 + 4.3264)\Big]$$

$$= \left[0.0004885\,\mu_1^2 - 0.0012286\,\mu_1\mu + 0.1143030\,\mu^2\right]$$

$$\mu_1\left[0.941915 + 0.578148\,(6.400\,\mu/\mu_1 - 2.08)\right] = -0.2553638\,\mu_1 + 3.6839347\,\mu$$

$$\mu_1\left[0.0025296 + 0.0047288\,(6.400\,\mu/\mu_1 - 2.08)\right] = -0.0073063\,\mu_1 + 0.0302643\,\mu$$

$$\mu_1\left[0.03750 + 0.0491275\,(6.400\,\mu/\mu_1 - 2.08)\right] = -0.0684352\,\mu_1 + 0.3144160\,\mu$$

$\underline{\underline{4\times 3}}$

$$\mu_1^3\Big[0.3266028 + 2.1771696\left(6.400\frac{\mu}{\mu_1} - 2.08\right) + 3.6179532\left(40.96\frac{\mu^2}{\mu_1^2} - 26.624\frac{\mu}{\mu_1} + 4.3264\right)$$

$$+ 1.701600\left(262.144\frac{\mu^3}{\mu_1^3} - 255.5904\frac{\mu^2}{\mu_1^2} + 83.06688\frac{\mu}{\mu_1} - 8.998912\right)\Big]$$

$$= -3.8617459\,\mu_1^3 + 58.9561025\,\mu_1^2\mu - 286.72126\,\mu_1\mu^2 + 446.06423\,\mu^3$$

$$\mu_1^2\Big[0.0056694 + 0.0055812\left(6.400\frac{\mu}{\mu_1} - 2.08\right) + 0.0014334\left(40.96\frac{\mu^2}{\mu_1^2} - 26.624\frac{\mu}{\mu_1} + 4.3264\right)\Big]$$

$$= \left[0.0002620\,\mu_1^2 - 0.0024432\,\mu_1\mu + 0.0582121\,\mu^2\right]$$

$$\mu_1\Big[0.5756148 + 0.3685396\left(6.4\frac{\mu}{\mu_1} - 2.08\right)\Big] = -0.2325426\,\mu_1 + 2.4806534\,\mu$$

$$\mu_1\Big[0.0042288 + 0.0067870\left(6.4\frac{\mu}{\mu_1} - 2.08\right)\Big] = -0.0093770\,\mu_1 + 0.0[illegible]\,\mu$$

$$\mu_1\Big[0.0491275 + 0.022056\left(6.4\frac{\mu}{\mu_1} - 2.08\right)\Big] = -0.0008130\,\mu_1 + [illegible]\,\mu$$

$$A_1\gamma^4 + B_1\gamma^2 + C_1 = 0$$

$$A_2\gamma^4 + B_2\gamma^2 + C_2 = 0$$

$$C_1 = +34413432\mu_1^3 - 48483172\mu_1^2\mu + 219.06117\mu_1\mu^2 - 316.14157\mu^3 + 0.2553638\mu_1 - 3.6839347\mu$$

$$B_1 = \left\{0.0004885\mu_1^2 - 0.0017226\mu_1\mu + 0.1143033\mu^2 - K(0.0084352\mu_1 - 0.3144160\mu)\right\}$$

$$A_1 = (0.0473063\mu_1 - 0.0302643\mu)$$

$$C_2 = +38[illegible]9\mu_1^3 - [illegible]9561025\mu_1^2\mu + 286.[illegible]126\mu_1\mu^2 - 444.06423\mu^3 + 0.2325426\mu_1 - 2.4866534\mu$$

$$B_2 = \left\{0.0002620\mu_1^2 - 0.0074432\mu_1\mu + 0.0287121\mu^2 - K(0.0088130\mu_1 - 0.178[illegible]\mu)\right\}$$

$$A_2 = (0.0093888\mu_1 - 0.0434368\mu)$$

The resultant for elimination of γ^2 is

$$(A_1C_2 - A_2C_1)^2 - (A_1B_2 - A_2B_1)(B_1C_2 - B_2C_1) = 0$$

$\underline{\mu = 0}$

$A_1 = 0.0073063\mu_1$	$A_2 = 0.0093882\mu_1$
$B_1 = 0.0004885\mu_1^2 - 0.0084352\mu_1 K$	$B_2 = 0.0002620\mu_1^2 - 0.0088130\mu_1 K$
$C_1 = 3.4413432\mu_1^3 + 0.2553638\mu_1$	$C_2 = 3.8617459\mu_1^3 + 0.2325826\mu_1$

$$(A_1C_2 - A_2C_1)^2 = \mu_1^4(-0.0040929\mu_1 - 0.0069834)^2$$

$$(A_1B_2 - A_2B_1) = \mu_1^2(-0.0000026719\mu_1 - 0.0000148009K)$$

$$(B_1C_2 - B_2C_1) = \mu_1^2[0.0039248\mu_1^3 + 0.0000466\mu_1 - (0.0022460\mu_1^2 - 0.0028894)K]$$

$$(40929\mu_1 + 6.9834)^2 + (0.26719\mu_1 + 1.48009K)\times$$
$$\times[0.9248\mu_1^3 + 0.04669\mu_1 - (2.2460\mu_1^2 - 0.28894)K] = 0$$

$$1675.183\mu_1^2 + 571.647\mu_1 + 48.7679$$

Let $\xi = \frac{\mu_2}{\mu}$

$$A_1 = 0.0073063\,\xi - 0.0302643$$

$$B_1 = \mu\,(0.0004885\,\xi^2 - 0.0017286\,\xi + 0.1143030) - K\,(0.0084352\,\xi - 0.3144160)$$

$$C_1 = \mu^2\,(3.4413432\,\xi^3 - 48.483172\,\xi^2 + 209.706117\,\xi - 316.14157) + (0.2553638\,\xi - 3.6031.7)$$

$$A_2 = 0.0093882\,\xi - 0.2434368$$

$$B_2 = \mu\,(0.0002620\,\xi^2 - 0.0024425\,\xi + 0.0587121) - K\,(0.0088130\,\xi - 0.1782784)$$

$$C_2 = \mu^2\,(3.8617459\,\xi^3 - 58.9561025\,\xi^2 + 286.72120\,\xi - 446.70413) + (0.2325476\,\xi - 2.444.534)$$

Let $\mu = 7$

$$A_1 = 0.0073063\,\xi - 0.0302643$$

$$B_1 = 0.0034195\,\xi^2 - 0.0121002\,\xi + 0.800121 - K\,(0.0084352\,\xi - 0.3144160)$$

$$C_1 = 168.62582\,\xi^3 - 2375.6754\,\xi^2 + 10233.99733\,\xi - 15490.93693 + 0.2553638\,\xi - 3.6031.7$$

$$C_1 = 168.625822\,\xi^3 - 2375.6754\,\xi^2 + 10234.252\,\xi - 15494.621$$

$$A_2 = 0.0093882\,\xi - 0.0434368$$

$$B_2 = 0.001834\,\xi^2 - 0.0171024\,\xi + 0.4109847 - K\,(0.0088130\,\xi - 0.1782784)$$

$$C_2 = 189.22555\,\xi^3 - 2888.8470\,\xi^2 + 14049.525\,\xi - 21859.634$$

We put

$A_1 = a_1 \qquad A_2 = a_2$

$B_1 = b_1 - c_1 K \qquad B_2 = b_2 - c_2 K$

$C_1 = d_1 \qquad C_2 = d_2$

$$(a_1 d_2 - a_2 d_1)^2 - [a_1 b_2 - a_2 b_1 - K(a_1 c_2 - a_2 c_1)][b_1 d_2 - b_2 d_1 - K(c_1 d_2 - c_2 d_1)] = 0$$

$$\therefore \{(a_1 d_2 - a_2 d_1)^2 - (a_1 b_2 - a_2 b_1)(b_1 d_2 - b_2 d_1)\}$$

$$+ \{(a_1 c_2 - a_2 c_1)(b_1 d_2 - b_2 d_1) + (a_1 b_2 - a_2 a_1)(c_1 d_2 - c_2 d_1)\} K$$

$$+ (a_1 c_2 - a_2 c_1)(c_1 d_2 - c_2 d_1) K^2 = 0$$

	ξ = −10		0		ξ = +10			
	1	2	1	2	1	2		
a	−0.1033273	−0.1373188	−0.0302643	−0.0434368	+0.0427987	+0.0504452		
b	+1.263073	+0.2154087	+0.800121	+0.4159877	+1.0210690	+0.4233627		
c	−0.398768	−0.2664084	−0.3144160	−0.1282284	−0.2302640	−0.0901484		
d	−529030.50	−140445.83	−15494.621	−21859.634	+22906.18	+18926.77		
$a_1d_2-a_2d_1$	−6448.228	1	−11.4702	1	−343.326	1		
$a_1b_2-a_2b_1$	+0.0943561	2	+0.0223165	2	−0.0333888	2		
$b_1d_2-b_2d_1$	−404030.71	3	−11122.300	3	+9677.015	3		
$a_1c_2-a_2c_1$	−0.0272311	4	−0.00826115	4	+0.0077474	4		
$c_1d_2-c_2d_1$	+114459.11	5	+4110.6624	5	−2300.916	5		
$(1)^2-(2)(3)$	41876096	6	379.776	6	118175.9	6		
$(3)(4)+(2)(5)$	21802.116	7	183.625	7	151.812	7		
$(4)(5)$	−3116.847	8	−33.9615	8	−17.8261	8		
$(7)/(8)$	−6.99493	9	−5.40689	9	−8.51628	9		
$(6)/(8)$	−13435.403	10	−11.18261	10	−6630.497	10		
$-(9)/2$			+2.70345	11	+4.25814	11		
$(11)^2-(10)$			+18.49125	12	+6648.63	12		
$(12)^{1/2}$			+4.30015	13	+81.5391	13		
$(11)+(13)$			+7.00360	K	+85.79726	K		

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	function f_1	function f_2	function f_3	$0.625 f_3$	$f_1 - 0.625 f_3$	W_1	$0.84375 f_3$	$f_2 - 0.84375 f_3$	W_2		
0	1.0000	1.00000	1.00000	0.6250000	0.3750000	1.000000	0.84375000	0.15625000	1.000000		
0.1	0.98010	0.998001	0.99980001	0.62487501	0.35522499	0.942267	0.84358126	0.15441974	0.988286		
0.2	0.92160	0.984064	0.99680256	0.62300160	+0.29859840	0.796262	0.84105216	0.14301184	0.915276		
0.3	0.82810	0.946729	0.98386561	0.61491601	+0.21318399	0.568491	0.83013661	0.11659239	0.746191		
0.4	0.70560	0.876096	0.94945536	0.59340960	+0.11219040	0.299174	0.80110296	0.07499304	0.479955		
0.5	0.56250	0.765625	0.87890625	0.54931641	+0.01318359	0.035156	0.74157715	0.02404785	0.153906		
0.6	0.40960	0.614656	0.75759616	0.47349760	-0.06389760	-0.170394	0.63922176	-0.02456576	-0.157221		
0.7	0.26010	0.431649	0.57744801	0.36090501	-0.10080501	-0.268813	0.48722176	-0.05557276	-0.355666		
0.8	0.12960	0.238144	0.34857216	0.21785760	-0.08825760	-0.235354	0.29410776	-0.05596376	-0.358168		
0.9	0.03610	0.073441	0.11826721	0.07391701	-0.03781701	-0.100845	0.09978796	-0.02634696	-0.168621		

The difficulty now is evidently that the curve form W_1, W_2 will not give a single-wave buckle.

With $\nu=0$, the equations on p. 383, become

$$q_2 + 0.666667\,S_2 - 0.083333 = \eta\left\{0.333333\,p_2 + 0 + 0.00123611\,f_1 - [illegible]\,f_2 - 0.00138889\,f_3\right\}$$

$$q_2 + 0 - 0.0833333 = \eta\left\{0.333333\,p_2 + 2a_2 - 0.02604167\,f_1 + 0.01845238\,f_2 + 0.02083333\,f_3\right\}$$

$$q_2 + 0.333333\,S_2 + 0.0833333 = \eta\left\{-0.33333\,p_2 - a_2 + 0.00868056\,f_1 - 0.01145595\,f_2 - 0.0111111\,f_3\right\}$$

$$q_2 + 2S_2 + 0.250000 = \eta\left\{-p_2 + 0 - 0.06770833\,f_1 - 0.07946429\,f_2 - 0.08750000\,f_3\right\}$$

$$q_2 + 0 - 0.250000 = \eta\left\{+p_2 + 3a_2 - 0.10520833\,f_1 + 0.06741071\,f_2 + 0.07500000\,f_3\right\}$$

$$0.66666667\,S_2 + 0 = \eta\left\{0 - 9a_2 + 0.02777778\,f_1 - 0.01875000\,f_2 - 0.[illegible]\,f_3\right\}$$

$$0.33333333\,S_2 + 0.16666667 = \eta\left\{-0.6666667\,p_2 - 3a_2 + 0.0347[illegible]\,f_1 - 0.03020833\,f_2 - 0.03194444\,f_3\right\}$$

$$1.66666667\,S_2 + 0.16666667 = \eta\left\{-0.666667\,p_2 + a_2 - 0.07638889\,f_1 - 0.06770834\,f_2 - 0.07638889\,f_3\right\}$$

$$2\,S_2 + 0.500000 = \eta\left\{-2p_2 - 3a_2 - 0.06250000\,f_1 - 0.14687500\,f_2 - 0.16250000\,f_3\right\}$$

$$S_2 + 0 = \eta\left\{0 - 3a_2 + 0.04166667\,f_1 - 0.02812500\,f_2 - 0.0333333\,f_3\right\}$$

$$S_2 + 0.50000 = \eta\left\{-9p_2 - 7[illegible]\,f_1 - 0.09062500\,f_2 - 0.09583333\,f_3\right\}$$

$$S_2 + 0.10000 = \eta\left\{-0.4p_2 + 0.6a_2 - 0.04583333\,f_1 - 0.04062500\,f_2 - 0.04583333\,f_3\right\}$$

$$S_2 + 0.25000 = \eta\left\{-p_2 - 1.5a_2 - 0.03125000\,f_1 - 0.07343750\,f_2 - 0.08125000\,f_3\right\}$$

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$$0.5000000 = \eta\left\{-2p_2 - 6h_2 + 0.06250000f_1 - 0.06250000f_2 - 0.06250000f_3\right\}$$

$$0.400000 = \eta\left\{-1.6p_2 - 9.6h_2 + 0.15000000f_1 - 0.05000000f_2 - 0.05000000f_3\right\}$$

$$0.150000 = \eta\left\{-0.6p_2 - 2.1h_2 + 0.01458333f_1 - 0.03281250f_2 - 0.03541667f_3\right\}$$

$$0.2500000 = \eta\left\{-p_2 - 3h_2 + 0.03125000f_1 - 0.03125000f_2 - 0.03125000f_3\right\}$$

$$0.250000 = \eta\left\{-p_2 - 6h_2 + 0.09375000f_1 - 0.03125000f_2 - 0.03125000f_3\right\}$$

$$0.250000 = \eta\left\{-p_2 - 3.5h_2 + 0.02430556f_1 - 0.05468750f_2 - 0.05902778f_3\right\}$$

$$3h_2 = 0.06250000f_1 + 0 + 0$$

$$2.5h_2 = 0.06944444f_1 + 0.02343750f_2 + 0.02777778f_3$$

$$3h_2 = 0.06250000f_1$$

$$3h_2 = 0.08333333f_1 + 0.02812500f_2 + 0.03333333f_3$$

$$0 = 0.02083333f_1 + 0.02812500f_2 + 0.03333333f_3$$

$$-f_3 = 0.625000f_1 + 0.843750f_2$$

Same thing !!!

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